Ph.D. Qualifying Examination (2005.9.23)
Algebra

Answer all the problems and show all your works.

1. (15%) Show that no group of order 48 is simple.

2. (10%) Let $H$ be a subgroup of a finite group $G$ with $[G : H] = p$, where $p$ is the smallest prime dividing the order of $G$. Prove that $H$ is normal in $G$. (Hint: Consider the action of $G$ on the coset of $H$.)

3. (10%) Let $R$ be a commutative Noetherian ring with identity. Show that $R[x]$ is also Noetherian.

4. (15%) Let $A \subseteq R$ be two integral domains containing identity such that $R$ is integral over $A$. Let $P$ and $Q$ be prime ideals in $R$ with $P \subseteq Q$. Show that $P = Q$ if $P \cap A = Q \cap A$.

5. (10%) Find all prime ideals in the ring $\mathbb{C}[x, y]/(xy - 1)$, where $\mathbb{C}$ is the field of all complex numbers.

6. (15%) Let $R$ and $S$ be two rings. Let $M$ be a right $R$-module, $N$ a right $S$-module and $P$ a $R$-$S$-bimodule with $R$ acting on the left and $S$ acting on the right. Show that there is an isomorphism of abelian groups from $\text{Hom}_S(M \otimes_R P, N)$ to $\text{Hom}_R(M, \text{Hom}_S(P, N))$.

7. (10%) Let $\mathbb{Z}_4$ is a cyclic group of order 4. We consider $\mathbb{Z}_4$ to be a $\mathbb{Z}$-module.
   (i) (5%) Find a projective $\mathbb{Z}$-module $P$ and a surjective $\mathbb{Z}$-homomorphism from $P$ to $\mathbb{Z}_4$.
   (ii) (5%) Find an injective $\mathbb{Z}$-module $J$ and an injective $\mathbb{Z}$-homomorphism from $\mathbb{Z}_4$ to $J$.

8. (15%) Let $E$ be a splitting field over $\mathbb{Q}$ of the equation $f(x) = x^4 - 5$, where $\mathbb{Q}$ is the field of all rational numbers.
   (i) (10%) Determine the Galois group of $E$ over $\mathbb{Q}$.
   (ii) (5%) Find all the intermediate fields $K$ between $E$ and $\mathbb{Q}$ satisfying $[E : K] = 2$. 