PhD Qualify Exam, Analysis, Feb. 24, 2006

Show all works

1. [10%] Explain the meaning of the Lebesgue integral $\int_R f(x) \, dx$. Begin by defining Lebesgue outer measure, measurable sets and measurable functions. Then explain how the Lebesgue integral is defined.

2. [15%] Suppose $f \in L^1(R)$. Let $F(x) = \int_R f(t) \frac{\sin xt}{t} \, dt$.
   
   a. Prove that $F$ is differentiable on $R$ and find $F'(x)$.
   
   b. Determine whether or not $F$ is absolutely continuous on every compact subinterval of $R$.

3. [15%] Suppose $f_n \in L^2[0,1], n = 1, 2, 3, \cdots$, and $\sum^\infty_1 \|f_n\|_2 < \infty$. Prove that
   
   a. $\sum^\infty_1 |f_n(x)| < \infty$ a.e.
   
   b. If $f(x) = \sum^\infty_1 f_n(x)$ a.e., then $f \in L^2[0,1]$ and $\|f\|_2 \leq \sum^\infty_1 \|f_n\|_2$.

4. [10%] Let $g$ be a nonnegative measurable function on $[0,1]$. Then

$$\log \int g(t) \, dt \geq \int \log(g(t)) \, dt$$

whenever the right side is defined.

5. [10%] Let $\langle A_n \rangle_{n \in N}$ be a sequence of connected subsets of the topological space $X$ such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in N$. Prove that the set $\bigcup_{n \in N} A_n$ is connected.

6. [15%] Let $(X, B, \mu)$ be a finite measure space. Suppose that $\nu$ is a measure defined on $B$ such that $\nu$ is absolutely continuous with respect to $\mu$ and such that $\nu(X)$ is finite. Let $g$ be the Radon-Nikodym derivative of $\nu$ with respect to $\mu$. Prove that

$$\int_X f \, d\nu = \int_X f \, g \, d\mu.$$
7. [15%] Let $f \in L^2[0, 1]$ and suppose that for each $q$ such that $1 < q < \infty$, $\left| \int_0^1 f g \right| \leq \|g\|_q$ for every $g \in C[0, 1]$.

a. Prove that $f \in \cap_{1 < p < \infty} L^p[0, 1]$.

b. Is $f \in L^\infty[0, 1]$? Explain your answer.

8. [10%] Find the Hausdorff dimension of $C \times C$, where $C$ is the Cantor Set, by computing the following quantities:

First

$$\lambda_\alpha^\epsilon(C \times C) = \inf \sum_{i=1}^{\infty} r_i^\alpha,$$

where $< r_i >$ are radii of sequence of balls $< B_i >$ that covers $C \times C$ and for which $r_i < \epsilon$.

Second,

$$m_\alpha(C \times C) = \lim_{\epsilon \to 0} \lambda_\alpha^\epsilon(C \times C).$$

Finally, Hausdorff dimension of $C \times C$ is $\inf\{\alpha : m_\alpha(C \times C) = \infty\}$. 