Klyachko Models for General Linear Groups of Rank 5 over a $p$-adic Field

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Abstract

This paper shows the existence and uniqueness of Klyachko models for irreducible unitary representations of $\text{GL}_5(\mathcal{F})$, where $\mathcal{F}$ is a $p$-adic field. It is an extension of the work of Heumos and Rallis on $\text{GL}_4(\mathcal{F})$.  

Keywords: Klyachko models, Whittaker-symplectic model
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1 Introduction

2 In 1984 [Kl], A. A. Klyachko started the investigation of a model, which we refer to as Klyachko models (also known as Whittaker-symplectic models) for $\text{GL}(n)$ over a finite field. This model consists of a series of representations $\mathcal{M}_{n, k}$, $0 \leq k \leq \left[\frac{n}{2}\right]$ with properties

1. Existence of models: Every irreducible representation of $\text{GL}(n, \mathbb{F}_q)$ is a subrepresentation of $\mathcal{M}_{n, k}$ for some $k$.

2. Uniqueness of models: For each irreducible representation, the multiplicity in $\mathcal{M}_{n, k}$ is at most one.

3. Disjointness of models: $\mathcal{M}_{n, i}$ and $\mathcal{M}_{n, j}$ are disjoint for $i \neq j$.

That is no irreducible representation can be embedded in both $\mathcal{M}_{n, i}$ and $\mathcal{M}_{n, j}$, for $i \neq j$.

These models generalize the usual Whittaker model by adding a symplectic component to the inducing subgroup. When $k = 0$, $\mathcal{M}_{n, 0}$ is the famous Whittaker model, a representation induced from a generic character on the unipotent radical of standard Borel subgroups of $\text{GL}_n$. When $n$ is even and $k = \frac{n}{2}$, $\mathcal{M}_{n, \frac{n}{2}}$ is induced from the trivial character of $\text{Sp}_n$ and is called symplectic model. The other “mixed” models $\mathcal{M}_{n, k}$, $0 < k < \left[\frac{n}{2}\right]$ are induced from characters of subgroups with smaller unipotent and symplectic components.

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\footnote{Refer to [HR] and [Kl] for notation and terminology.
Then followed by Michael J. Heumos and Stephen Rallis ([HR], 1990) who consider the realization of these models on $GL_n$ over $p$-adic fields. At first, like finite fields case disjointness and uniqueness of this model are expected for all irreducible representations, but soon later they found an irreducible non-unitary representation of $GL_2(F)$ which does not have any one of such models. Then they restricted the discussion to irreducible unitary representations, and proved the uniqueness of the symplectic models and the disjointness for unitary representations of the different models. Moreover, for $n \leq 4$ they proved that any unitary irreducible representation admits a unique Klyachko model. Following their work, a unique model is explicitly classified for each irreducible unitary representation of $GL_2$ in Theorem 5.7.

In [OS1], O. Offen and E. Sayag showed that a certain family of irreducible, unitary representations of $GL_{2n}$ have symplectic models by embedding a local problem into a global setting. Recently, they further prove that every irreducible, unitary representation of $GL_n$ admits a nontrivial embedding as a submodule of $M_k$ model for a unique $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ ([OS2]).

2 Notation and Terminology

For notation and terminology, we follow [HR], [BZ1] and [BZ2]. Throughout, $F$ denotes a $p$-adic field and $G_n$ denotes $GL(n, F)$.

The standard (upper triangular) parabolic subgroups of $G_n$ are parameterized by ordered partitions $(n_1, ..., n_k)$ of $n = n_1 + ... + n_k$. Let $P_{n_1, ..., n_k}$ denote the associated parabolic subgroups and $N_{n_1, ..., n_k}$ denote its unipotent radical, $J_n$ denotes the $2n \times 2n$ matrix \[
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\] and the associated symplectic form $\langle x, y \rangle = \langle xJ_n y \rangle$ is denoted by $J$. The symplectic groups $Sp_{2n}$ are the set of elements preserving the form $J$ in $G_{2n}$. Let $U_n$ denote the group of upper triangular unipotent matrices in $G_n$. With $U_{n-2k}$ embedded in the upper left and $Sp_{2k}$ in the lower right, let

$$M_k = (U_{n-2k} \times Sp_{2k})N_k,$$

where $N_k = N_{n-2k, 2k}$.

We denote $\nu$ the homomorphism $g \mapsto |\det g|$ and $\delta_p$ the modular function of the group $P$. A character (one-dimensional representation) of $G_n$ is in the form of $g \mapsto \chi(\det g)$ for some character $\chi$ of $F^\times$. We sometimes write $\chi_n$ to indicate the group $G_n$ involved. Induction is always normalized unless otherwise state, with Ind and Ind respectively) denoting full (compact respectively) induction. Given representations $\sigma_i$ of $G_{n_i}$, $i = 1, ..., k$, extend $\sigma_{n_1} \otimes ... \otimes \sigma_{n_k}$.
to $P_{n_1,...,n_k}$, so that it is trivial on $N_{n_1,...,n_k}$. Denote
\[
\text{Ind}_{P_{n_1,...,n_k}}^{G_{n_1+...+n_k}} \sigma_{n_1} \otimes \cdots \otimes \sigma_{n_k} \text{ by } \sigma_{n_1} \times \cdots \times \sigma_{n_k}.
\]

Given a unipotent radical $N_{n_1,...,n_k}$ and a representations $\pi$ of $G_n$, the **Jacquet functor** $r_{n_1,...,n_k}$ is defined to be the functor mapping $\pi$ to the quotient space
\[
V_\pi / \{ \pi(n)v - v | v \in V_\pi, n \in N_{n_1,...,n_k} \}.
\]
The quotient space is a $G_{n_1} \times \cdots \times G_{n_k}$ module and is called the **Jacquet module** of $\pi$. Let $\tilde{r}$ denote the normalized **Jacquet functor** (refer to [BZ2]). Let $\psi$ be any nontrivial, complex, additive character of $F$. Define the character $\psi_n$ of $U_n$ by
\[
\psi_n(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}), u = (u_{i,j}).
\]

A **generic** (or **nondegenerate**, **Whittaker**) character is a character which is nontrivial on all the simple root groups in $U_n$. For $1 \leq k \leq \lceil \frac{n}{2} \rceil$, define a series of models for $G_n$ to be representations
\[
M_{n,k} = \text{Ind}_{M_k}^{G_n} (\psi_{n-2k} \otimes 1 \otimes 1).
\]

Denote $\psi_{n-2k} \otimes 1 \otimes 1$ by $\psi_{n-2k}$. When $n$ is understood, we simply write $M_k$. We call $M_0$ **Whittaker model**. The Whittaker models for any two Whittaker characters are equivalent, since the diagonal torus of $G_n$ normalizes $U_n$ and acts transitively on the set of Whittaker characters. The Weyl group of $G_n$ is the symmetric group $S_n$, and we use cycle forms $(i_1, i_2, \ldots, i_k)$ of permutations to denote the corresponding Weyl elements in $W$. For example, in $G_4$,
\[
(1, 2, 3) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We also denote
\[
\text{Ind}_{n_1,...,n_k}^{G_{n_1+...+n_k}} = \text{Ind}_{G_{n_1} \times \cdots \times G_{n_k}}^{G_{n_1+...+n_k}}, r_{n_1,...,n_k} = r_{G_{n_1} \times \cdots \times G_{n_k}},
\]
and
\[
\text{Hom}_{n_1,...,n_k} = \text{Hom}_{G_{n_1} \times \cdots \times G_{n_k}}.
\]

### 3 Known Results on $\text{GL}_n$

Denote by $\text{AlgG}$ the set of all smooth representations of an algebraic group $G$. 

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Proposition 3.1. (Prop 1.9, [BZ2]) Let $M, U$ be closed subgroups of $G_n$ such that $M$ normalizes $U$, $M \cap U = \{e\}$ and the subgroup $P = MU \subset G_n$ is closed. Then

1. The functors $\text{Ind}_P$, $\text{ind}_P$ are exact.
2. The functor $\tilde{\tau}_M$ is left adjoint to $\text{Ind}_P$. That is
   $$\text{Hom}_M(\tilde{\tau}_M(\pi), \rho) \simeq \text{Hom}_{G_n}(\pi, \text{Ind}_P \rho).$$

3. Induction by stages: Let $S, T$ be subgroups of $M$ and $H = ST$ such that the functors $\text{Ind}_H$, $\text{ind}_H : \text{Alg}S \mapsto \text{Alg}M$ and $\tilde{\tau}_S : \text{Alg}M \mapsto \text{Alg}S$ are well defined. Then
   $$\text{ind}_{P_H} \circ \text{ind}_H^M = \text{ind}_{P_H}^G, \quad \text{ind}_H^M \circ \text{ind}_H^M = \text{Ind}_H^G, \quad \tilde{\tau}_S \circ \tilde{\tau}_M = \tilde{\tau}_{ST}.$$ 

Theorem 3.2. (Jacquet’s Theorem, [BZ1]) Let $\pi \in \text{Alg}G_n$ be irreducible. Then there exists a parabolic triple $(P, M, U)$ of $G_n$ and an irreducible cuspidal representation $\rho \in \text{Alg}M$ such that $\pi$ can be embedded into $\text{ind}_{P,H}^G(\rho)$. In particular, $\pi$ is admissible.

Let $\alpha = (n_1, ..., n_r)$ be an ordered partition of $n$, and let $G_\alpha = G_{n_1} \times \cdots \times G_{n_r}$ be the subgroup of $G_n$ embedded as the subgroup of block-diagonal matrices. By blocks of $\alpha$ we mean the sets of indices $I_1 = \{1, ..., n_1\}, I_2 = \{n_1 + 1, ..., n_1 + n_2\}, ..., I_r = \{n_1 + ... + n_{r-1} + 1, ..., n\}$.

For two partitions $\beta, \gamma$ with blocks $I_1, ..., I_r$ and $J_1, ..., J_s$ respectively, set

$$W^{\beta, \gamma} = \{w \in W | w(k) < w(l) \text{ if } k < l \text{ and } k, l \text{ belong to the same } I_i \},$$

$$w^{-1}(k) < w^{-1}(l) \text{ if } k < l \text{ and both } k \text{ and } l \text{ belong to the same } J_j \}.

Let $F_w = \text{ind}_{\gamma w^w}^\beta \circ w \circ \tilde{\tau}_{\beta w^{-1}(\gamma)}^\beta$.

Theorem 3.3. (Thm 1.2, [Ze]) The functor

$$F = \tilde{\tau}_\beta \circ \text{ind}_\beta^n : \text{Alg}G_\beta \mapsto \text{Alg}G_\gamma$$

is glued together from those $F_w$ where $w \in W^{\beta, \gamma}$. That is for $\pi$ a representation of $G_\alpha$ and $\beta, \gamma$ partitions of $n$ the set of composition factors of $\tilde{\tau}_\beta \circ \text{ind}_\beta^n(\pi)$ is $\{F_w(\pi) | w \in W^{\beta, \gamma}\}$.

In the following, $C_c^\infty(X)$ denotes the space of smooth, compactly supported functions on a $p$-adic space $X$, and $\mathfrak{D}(X)$ denotes the space of complex-valued linear functionals on $C_c^\infty(X)$. Elements of $\mathfrak{D}(X)$ are called distributions. Given a Lie group $G$, define the left and right translations $l_g$ and $r_g$ on $G; C_c^\infty(G)$ and $\mathfrak{D}(G)$ as the following:

$$l_g \cdot x = gx; \quad r_g \cdot x = xg^{-1};$$

$$(l_g \cdot f)(x) = f(g^{-1}x); \quad (r_g \cdot f)(x) = f(xg);$$

$$(l_g \cdot T)(f) = T(l_{g^{-1}} \cdot f); \quad (r_g \cdot T)(f) = T(r_{g^{-1}} \cdot f),$$

where $g, x \in G; f \in C_c^\infty(G)$ and $T \in \mathfrak{D}(G)$.

If $G$ acts on a $p$-adic space $X$, we define the action of $l_g, g \in G$ on $X, C_c^\infty(X)$ and $\mathfrak{D}(G)$ in a similar manner.
Lemma 3.4. (Bernstein’s localization principle, Theorem 6.9, [BZ1]) Assume that a p-adic group $G$ acts on a p-adic space $X$ by $q : X \mapsto X$ constructively, which means that the graph $\{(x,gx) | g \in G, x \in X\}$ of $G$ is the union of finitely many locally closed subsets of $X \times X$. If every fiber $X_y = q^{-1}(y)$ is $G$-invariant and if $\mathcal{D}(X_y)^G = 0$ for every $y \in X$, then $\mathcal{D}(X)^G = 0$.

A segment $\triangle$ is a representation of $G_n$ of the form of

$$\rho \times \nu\rho \times \ldots \times \nu^k \rho,$$

where $k \in \mathbb{N}$, $mk = n$ and $\rho$ is an irreducible cuspidal representation of $G_n$. We write $\triangle = [\rho, \nu^k \rho]$ to indicate the beginning and the end of a segment. Two segments $\triangle_1, \triangle_2$ are linked if $\triangle_1 \not\subset \triangle_2, \triangle_2 \not\subset \triangle_1$, and $\triangle_1 \cup \triangle_2$ is also a segment. Let $\triangle_1 = [\rho_1, \nu^m \rho_1], \triangle_2 = [\rho_2, \nu^m \rho_2]$. $\triangle_1$ precedes $\triangle_2$ if $\triangle_1$ and $\triangle_2$ are linked and $\rho_2 = \nu^m \rho_1$ for some $m > 0$.

If $\pi$ is a representation, we denote by $\langle \pi \rangle$ (respectively $L(\pi)$) the unique irreducible submodule (respectively the unique irreducible quotient module) of $\pi$, when it exists. A sufficient condition for existence is explained in the following theorem and in many useful cases unique submodules and unique quotients do exist.

Theorem 3.5. (Thm 6.1, [BZ2]) Let $\triangle_1, \ldots, \triangle_r$ be segments of $G_n$ such that for each pair of indices $i < j$, $\triangle_i$ does not precede $\triangle_j$. Let the same condition hold for segments $\triangle'_1, \ldots, \triangle'_r$. Then

1. The representation $\langle \triangle_1 \rangle \times \ldots \times \langle \triangle_r \rangle$ has a unique irreducible submodule, denoted by $\langle \triangle_1, \ldots, \triangle_r \rangle$.

2. $\langle \triangle_1, \ldots, \triangle_r \rangle$ and $\langle \triangle'_1, \ldots, \triangle'_r \rangle$ are isomorphic if and only if the sequences of segments $\{\triangle_1, \ldots, \triangle_r\}$ and $\{\triangle'_1, \ldots, \triangle'_r\}$ are equal up to a chain of transpositions of two non-linked neighbors.

3. Any irreducible representation in $\text{Alg}G_n$ is isomorphic to some representation of the form $\langle \triangle_1, \ldots, \triangle_r \rangle$.

4. The representation $\langle \triangle_1 \rangle \times \ldots \times \langle \triangle_r \rangle$ is irreducible if and only if $\triangle_i$ and $\triangle_j$ are not linked for each pair $i, j = 1, \ldots, r$.

Theorem 3.6. (Thm 1.2.5, [Kui]) Let $\triangle_1, \ldots, \triangle_r$ be segments of $G_n$ such that for each pair of indices $i < j$, $\triangle_i$ does not precede $\triangle_j$. Let the same condition hold for segments $\triangle'_1, \ldots, \triangle'_r$. Then

1. The representation $L(\triangle_1) \times \ldots \times L(\triangle_r)$ has a unique irreducible quotient, denoted by $L(\triangle_1, \ldots, \triangle_r)$.

2. $L(\triangle_1, \ldots, \triangle_r)$ and $L(\triangle'_1, \ldots, \triangle'_r)$ are isomorphic if and only if the sequences of segments $\{\triangle_1, \ldots, \triangle_r\}$ and $\{\triangle'_1, \ldots, \triangle'_r\}$ are equal up to a chain of transpositions of two non-linked neighbors.

3. Any irreducible representation in $\text{Alg}G_n$ is isomorphic to some representation of the form $L(\triangle_1, \ldots, \triangle_r)$.
4. The representation \( L(\triangle_1) \times \ldots \times L(\triangle_r) \) is irreducible if and only if \( \triangle_i \) and \( \triangle_j \) are not linked for each pair \( i, j = 1, \ldots, r \).

**Theorem 3.7.** (Thm 9.3, [Ze]) An irreducible representation \( \pi \) in \( \text{AlgG}_n \) is quasi-square-integrable if and only if it is isomorphic to \( L(\triangle) \) for some segment \( \triangle = [\rho, \nu^m \rho] \), where \( \rho \) is an irreducible cuspidal representation of \( G_k \), \( km = n, k, m \in \mathbb{N} \). That is, it is the unique quotient of some segment. In particular, every irreducible cuspidal representation of \( G_n \) is quasi-square-integrable.

**Lemma 3.8.** (Lem 3.2, [Ta1]) Let \( \triangle_1, \triangle_2 \) be two segments.

1. If \( \triangle_1 \) and \( \triangle_2 \) are not linked, then
   \[ L(\triangle_1) \times L(\triangle_2) = L(\triangle_1, \triangle_2). \]

2. If \( \triangle_1 \) and \( \triangle_2 \) are linked, then
   \[ L(\triangle_1) \times L(\triangle_2) = L(\triangle_1, \triangle_2) + L((\triangle_1 \cap \triangle_2), (\triangle_1 \cup \triangle_2)). \]

   where the summation is in the sense of semi-simplification (i.e. \( L(\triangle_1, \triangle_2) \) and \( L((\triangle_1 \cap \triangle_2), (\triangle_1 \cup \triangle_2)) \) are composition factors of \( L(\triangle_1) \times L(\triangle_2) \)).

**Theorem 3.9.** (Thm 9.7, [Ze])

1. For any \( k \) segments \( \triangle_1, \ldots, \triangle_k \), the representation
   \[ \pi = L(\triangle_1) \times \ldots \times L(\triangle_k); \]
   has a nontrivial Whittaker functional. In particular, every irreducible quasi-square-integrable representation of \( G_n \) is generic.

2. Any generic representation \( \pi \) of \( \text{AlgG}_n \) can be decomposed as a product
   \[ \pi = L(\triangle_1) \times \ldots \times L(\triangle_k), \]
   for some segments \( \triangle_1, \ldots, \triangle_k \), such that no two of them are linked. Moreover the set \( \{\triangle_1, \ldots, \triangle_k\} \) is uniquely determined by \( \pi \) up to isomorphisms of representations.

**Theorem 3.10.** (Thm 2, [Ro]) Let \( \pi_i \) be irreducible representations of \( G_n \), \( i = 1, \ldots, k \), and \( n = n_1 + \ldots + n_k \). Then
\[
\text{Hom}_n(\pi_1 \times \ldots \times \pi_n, M_{n,0}) \cong \text{Hom}_{n_1,\ldots,n_k}(\pi_1 \otimes \ldots \otimes \pi_n, M_{n_1,0} \otimes \ldots \otimes M_{n_k,0}).
\]

Now we recall the classification of irreducible unitary representations of \( G_n \) due to M. Tadić [Ta2].

Let \( D_0(n) \) denote the set of isomorphism classes of irreducible representations of \( G_n \) which are **square-integrable modulo the center** and \( D_0 = \bigcup_{n \geq n} D_0(n) \). Let \( D(n) \) be the set of representations of
the form $\nu^n \delta$, where $\alpha$ is real and $\delta \in D_0(n); D = \bigcup_{n \geq 0} D(n), M(D)$ is the collection of all finite (unordered) multisets on $D$.

For $\rho$ an irreducible cuspidal representation and $n \in \mathbb{N}$, let

$$\triangledown[n]^\rho = \nu_{-\alpha - \frac{\alpha + 1}{2}} \rho \times \nu_{-\alpha - \frac{\alpha + 2}{2}} \rho \times \ldots \times \nu_{-\alpha - \frac{\alpha + n}{2}} \rho.$$ 

That is, $\triangledown[n]^\rho$ is a segment with exponents of $\nu$ symmetric around 0.

Given $a = \langle \delta_1, \ldots, \delta_n \rangle \in M(D), \delta_i = \nu^{\alpha_i} \delta_0^i, \delta_0^i \in D_0$, we may assume that $\alpha_1 \geq \ldots \geq \alpha_n$. The induced representation $\delta_1 \times \ldots \times \delta_n$ has a unique irreducible quotient $L(a)$.

Given an irreducible representation $\sigma$, let $\sigma^+$ denote its Hermitian (complex conjugate) contragradient. Set $\langle \sigma, \alpha \rangle = \nu^{\alpha} \sigma \times \nu^{-\alpha} \sigma^+$, for $\alpha$ positive. For a positive integer $n$ and $\delta \in D_0$, set $u(\delta, n) = L(\nu^{\rho} \delta \times \nu^{\rho - 1} \delta \times \ldots \times \nu^{-\rho} \delta)$, where $p = \frac{n-1}{2}$. Thus if $\delta$ is a representation of $G_m, u(\delta, n)$ is a representation of $G_{nm}$. We sometimes write $u(\delta_m, n)$ to emphasize the rank of $\delta$.

**Theorem 3.11.** (Thm 7.5, [Ta2]) Let

$$\mathfrak{B} = \left\{ u(\delta, n), \prod_{n \geq 0} u(\delta(n), \alpha) | \delta \in D_0, 0 < \alpha < \frac{1}{2} \right\};$$

$$a(n, d)^\rho = \langle \nu_{-\frac{1}{2} - \delta} \triangledown[d]^\rho, \nu_{-\frac{1}{2} - \delta} \triangledown[d]^\rho, \ldots, \nu_{-\frac{1}{2} - \delta} \triangledown[d]^\rho \rangle,$$

where $n, d \in \mathbb{N}$, and $\rho$ is an irreducible cuspidal representation. Then

1. If $\sigma_1, \ldots, \sigma_r \in \mathfrak{B}$, then $\sigma_1 \times \ldots \times \sigma_r$ is irreducible and unitary.
2. If $\pi$ is an irreducible unitarizable representation, then there exist $\tau_1, \ldots, \tau_s \in \mathfrak{B}$, unique up to permutations, such that $\pi = \tau_1 \times \ldots \times \tau_s$.
3. $L(a(n, d)^\rho) = \langle a(d, n)^\rho \rangle = u(\delta(\rho, d), n)$, where $\delta(\rho, d) = L(\triangledown[d]^\rho)$.

For this part of notation and results, we refer to [KV] and [Wa]. Let $G$ denote a unimodular $p$-adic group.

**Definition 3.12.** Let Frechet spaces $V_{\pi_1}, W_{\pi_2}$ be representations of $G$. A separately continuous bilinear form

$$B : V_{\pi_1} \times W_{\pi_2} \rightarrow \mathbb{C}$$

is said to be a $(\pi_1, \pi_2)$-intertwining form if

$$B \circ (\pi_1 \otimes \pi_2)(g) = B, g \in G.$$ 

We denote the linear space of these forms by $I(\pi_1, \pi_2)$.

If $W_{\pi_2}$ is unitary with the inner product $<, >_{\pi_2}$, we define

$$B_T(v, w) = < T v, w >_{\pi_2}, v \in V_{\pi_1}, w \in W_{\pi_2}.$$ 

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for any given intertwining operator $T \in \text{Hom}_G(V_{\pi_1}, W_{\pi_2})$. Then $B_T \in I(\pi_1, \pi_2)$, and

$$\dim \text{Hom}_G(V_{\pi_1}, W_{\pi_2}) \leq \dim I(\pi_1, \pi_2).$$

**Theorem 3.13.** (Thm 4.7, [KV]) Assume that $G$ is a unimodular $p$-adic group and $R, Q$ are closed subgroups of $G$. Let $\pi_1 = \text{ind}_{G}^{G} \chi_1$, $\pi_2 = \text{ind}_{Q}^{G} \chi_2$, where $\chi_1$ (respectively $\chi_2$) is a character of $R$ (respectively $Q$). Then there exists a linear isomorphism between the linear space $I(\pi_1, \pi_2)$ and the linear space $\mathcal{D}(G)^{R \times Q}$ of $R \times Q$-invariant distributions on $G$. Here the $R \times Q$-action on $T \in \mathcal{D}(G)$ is given by

$$\langle r, q \rangle \cdot T = (\delta_R(r)\delta_Q(q))^{-1}\chi_1^{-1}(r)\chi_2(q)(l_r \circ r_q) \cdot T,$$

for $r \in R, q \in Q$.

**Proposition 3.14.** Assume that $G$ is a unimodular $p$-adic group, and $R, Q$ are closed-unimodular subgroups of $G$. Let $\chi_1$ (respectively $\chi_2$) be a character of $R$ (respectively $Q$). Then

$$\dim \text{Hom}_G(\text{Ind}_Q^{G} \chi_2^{-1}, \text{Ind}_R^{G} \chi_1^{-1}) \leq \dim \mathcal{D}(G)^{R \times Q}.$$

**Proof.** The result follows the above theorem and the facts:

1. $\text{Hom}_G(\text{Ind}_Q^{G} \chi_2^{-1}, \text{Ind}_R^{G} \chi_1^{-1}) \cong \text{Hom}_G(\text{Ind}_R^{G} \chi_1^{-1}, \text{Ind}_Q^{G} \chi_2^{-1}).$

2. $\text{Ind}_R^{G} \chi_1^{-1} \cong \text{ind}_G^{G} \chi_1$, when $R$ is unimodular (refer to [BZ1], 2.29).

## 4 Known Results on Klyachko Models

For $G_2$, there are only three types of irreducible representations: cuspidal representations, submodules of segments $[\rho, \nu \rho]$, and quotients of segments $[\rho, \nu \rho]$, where $\rho$ is an irreducible cuspidal representation on $G_1 = F^*$. Submodules of segments are in fact characters, and hence have symplectic models (refer to Lemma 5.5). Cuspidal representations and quotients of segments all satisfy the criterion of Theorem 3.7, and admit Whittaker models. Therefore for $G_2$ every irreducible representation has either a Whittaker or a symplectic model.

**Theorem 4.1.** (Thm 2.4.2, [HR]) Let $\pi$ be an irreducible representation of $G_{2n}$, then $\dim \text{Hom}_G(\pi, \mathcal{M}_n) \leq 1$.

**Theorem 4.2.** (Thm 3.2.2, [HR]) An irreducible representation of $G_n$ cannot have both a Whittaker model and a symplectic model.
Theorem 4.3. (Thm 3.1, [HR]) Let \( \pi \) be an irreducible unitary representation of \( G_n \). Then \( \text{Hom}(\pi, \mathcal{M}_i) \) is nonzero for at most one integer \( i \), \( 0 \leq i \leq \lfloor \frac{n}{2} \rfloor \).

Theorem 4.4. (Thm 9.1.1, [HR]) Let \( I = \text{Ind}_{2,1}^1(1 \times \nu) \otimes \nu^{-1} \). The representation \( (I) \) (unique irreducible submodule of \( I \)) has neither a Whittaker model nor a mixed model.

Theorem 4.5. (Thm 8.1, [HR]) Let \( \pi \) be an irreducible, unitary representation of \( G_3 \). Then \( \pi \) can be uniquely embedded as a submodule of Whittaker model \( \mathcal{M}_0 \) or mixed model \( \mathcal{M}_1 \).

Theorem 4.6. (Thm 11.5, [HR]) Let \( \pi \) be an irreducible, unitary representation of \( G_4 \). Then \( \pi \) can be uniquely embedded as a submodule of Whittaker model \( \mathcal{M}_0 \), mixed model \( \mathcal{M}_1 \) or symplectic model \( \mathcal{M}_2 \).

Theorem 4.7. ([OS1]) Let \( \pi = \sigma_1 \times \cdots \times \sigma_l \times \tau_{l+1} \times \cdots \times \tau_s \) be a unitary irreducible representation of \( GL_2n(\mathbb{F}) \), with \( \sigma_i = u(\delta_{k_i}, 2m_i) \in \mathfrak{B} \) and \( \tau_i = \prod (u(\delta_{k_i}, 2m_i), \alpha_i) \in \mathfrak{B} \). Then \( \pi \) admits a symplectic model.

Theorem 4.8. (Theorem 2, [OS2]) Let \( \pi \) be any irreducible, unitary representation of \( GL_n(\mathbb{F}) \). Then there exists a unique \( k = k(\pi) \), \( 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \) such that \( \pi \) admits an embedding as a submodule of \( \mathcal{M}_k \) model.

5 Klyachko Models on \( GL_5 \)

Lemma 5.1. For \( i \neq j \), \( \text{Hom}_{G_4}(\mathcal{M}_i, \mathcal{M}_j) = 0 \).

Proof. By Proposition 3.14, it suffices to show the following claim: If a distribution \( T \) on \( G_4 \) satisfying

\[
\psi_{4-2i}(m_i)\psi_{4-2j}^{-1}(m_j)T((l_{m_i} \circ r_{m_j}) \cdot f) = T(f), \quad (5.1),
\]

\[ f \in C_c(G_4), \ m_i \in M_i, \ m_j \in M_j, \ i \neq j. \]

then \( T \) is trivial.

First note that all \( M_i \), \( i = 0, 1, 2 \) involved here are unimodular.

Let \( H = M_i \times M_j \), for \( i \neq j \). The action of \( H \) on \( G_4 \) is given by

\[ (m_i, m_j) \cdot g = m_i gm_j^{-1} \] for \( (m_i, m_j) \in H, \ g \in G_4 \).

This action is constructive by Theorem A in 6.15 of [BZ1]. Then by Lemma 3.4 Bernstein’s localization principle, it is enough to show that \( T_x \) is trivial for all \( x \in M_i \setminus G_4/M_j \), where \( T_x \) is a distribution on \( M_i \times M_j \) satisfying equation (5.1).

Define a character \( \psi_H \) on \( H \) by

\[ \psi_H(m_i, m_j) = \psi_{4-2i}(m_i)\psi_{4-2j}^{-1}(m_j) \] for \( (m_i, m_j) \in H \).
and the action of \((m_i, m_j) \in H\) on \(C_c^\infty(G_4)\) by

\[
(m_1, m_2) \cdot \eta(g) = \psi_{H}^{-1}(m_1^{-1}, m_2) \eta(m_1^{-1} gm_2), \quad \text{for } \eta \in C_c^\infty(G_4).
\]

Let \(T_x\) be a nonzero \(H\)-invariant distribution on an \(H\)-orbit \(Y_x = M_i x M_j\), i.e.

\[
T_x((m_i, m_j) \cdot \eta) = \psi_{H}^{-1}(m_1^{-1}, m_2) T_x((l_{m_i} \circ r_{m_j}) \cdot \eta) = T_x(\eta)
\]

for \((m_i, m_j) \in H\) and \(\eta \in C_c^\infty(Y_x)\). Equivalently, \(T_x\) satisfies equation (5.1). Let \(H_x\) denote the stabilizer of \(x\) in \(H\). Then \(Y_x \cong H/H_x\). Note that \(C_c^\infty(Y_x) \cong \text{ind}_{H_x}^H 1\) (un-normalized compact induction), and

\[
T_x \in \text{Hom}_H(\text{ind}_{H_x}^H 1, \psi_H) \cong \text{Hom}_{H_x}(\delta_H \delta_{H_x}^{-1}, \text{Res}_{H_x} \psi_H)
\]

by Frobenius reciprocity, where \(\delta_H\) (respectively \(\delta_{H_x}\)) is the modular function of \(H\) (respectively \(H_x\)). Since the absolute value of \(\psi_H \equiv 1\) and \(\delta_H \delta_{H_x}^{-1}\) is positive, by Schur’s Lemma we have

\[
\dim \text{Hom}_{H_x}(\delta_H \delta_{H_x}^{-1}, \text{Res}_{H_x} \psi_H) = 0 \quad \text{or} \quad \delta_H \delta_{H_x}^{-1} = \text{Res}_{H_x} \psi_H \equiv 1.
\]

Proposition 1.3 in [Kl] shows that there are no admissible double cosets between \((M_i, \xi_{4-2i})\) and \((M_j, \xi_{4-2j})\), so \(\text{Res}_{H_x} \psi_H \neq 1\) and

\[
\text{Hom}_{H_x}(\delta_H \delta_{H_x}^{-1}, \text{Res}_{H_x} \psi_H) = 0
\]

for all \(x \in M_i \backslash G_4 / M_j\). Therefore \(\mathfrak{O}(G_4)^H = 0\) and

\[
\text{Hom}_{G_4}(M_i, M_j) = 0
\]

follows. \(\square\)

**Theorem 5.2.** \(\mathcal{M}_{4,i}\) and \(\mathcal{M}_{4,j}\) are disjoint for \(i \neq j\). That is an irreducible representation of \(G_4\) cannot have both (nontrivial) \(M_i\) model and \(M_j\) model for \(i \neq j\).

**Proof.** Although the proof is routine (refer to Theorem 3.1 and Theorem 3.2.2, [HR]), we repeat it for the sake of completion. \(\mathcal{M}_0\) and \(\mathcal{M}_2\) are disjoint by Theorem 4.2 and it remains to show that \(\mathcal{M}_t\) and \(\mathcal{M}_{1}\) (respectively, \(\mathcal{M}_1\) and \(\mathcal{M}_2\)) are disjoint. Let \(\pi\) be an irreducible representation of \(G_4\). Assume that \(\pi\) has both nontrivial \(\mathcal{M}_0\) (Whittaker) model and \(\mathcal{M}_{1}\) model. By Proposition 3.2.1 of [HR], the contragradient \(\tilde{\pi}\) of \(\pi\) also admits a Whittaker model. The dual of \(\text{Hom}_{G_4}(\tilde{\pi}, \mathcal{M}_0) \neq 0\) gives \(\text{Hom}_{G_4}(\text{ind}_{M_0}^{G_4} \psi_{\tilde{\pi}}^{-1}, \pi) \neq 0\) (refer to [GK] or [BZ1]). The composition of nontrivial

\[
T_1 \in \text{Hom}_{G_4}(\text{ind}_{M_0}^{G_4} \psi_{\tilde{\pi}}^{-1}, \pi) \quad \text{and} \quad T_2 \in \text{Hom}_{G_4}(\pi, M_1)
\]
produces a nontrivial intertwining operator $T$ (since $\pi$ is irreducible) in
\[ \text{Hom}_{G_4}(\text{ind}_{M_0 \cap \psi_4^{-1}}^G \psi_4^{-1}, M_1). \]

The right action of $M_1$ on $M_0 \setminus G_4$ is constructive by Theorem A, 6.15, [BZ1]. The restriction of $T$ to the coset $M_0 w M_1$ is associated with $\text{ind}_{M_1 \cap w^{-1} M_0 w}^M \psi_4^{-w}$, where $\psi_4^{-w}(g) = \psi_4^{-1}(w gw^{-1})$, for $g \in M_1 \cap w^{-1} M_0 w$. Frobenius reciprocity gives
\[ \text{Hom}_{M_1}(\text{ind}_{M_1 \cap w^{-1} M_0 w}^M \psi_4^{-w}, \psi_2) \cong \text{Hom}_{M_1 \cap w^{-1} M_0 w}(\psi_4^{-w}, \psi_2). \]

By the result of [Kl], there exists no admissible double cosets of between $(M_0, \tilde{\psi}_4^{-1})$ and $(M_1, \tilde{\psi}_2)$ and $\text{Hom}_{M_1 \cap w^{-1} M_0 w}(\psi_4^{-w}, \psi_2) = 0$ for all $w \in G_4$. Hence by Bernstein’s localization principle
\[ \text{Hom}_{G_4}(\text{ind}_{M_0 \cap \psi_4^{-1}}^G \psi_4^{-1}, \text{Ind}_{G_4}^G \psi_2) = 0, \]
which contradicts to our assumption and therefore $\pi$ cannot possess both $M_0$ model and $M_1$ model. The proof for disjointness of $M_1$ and $M_2$ follows the same argument, since $\tilde{\pi}$ also admits a symplectic model if $\pi$ does (Also refer to Proposition 3.2.1 of [HR]).

\[ \square \]

**Lemma 5.3.** For $k \neq \pm 2, \rho, \tau$ unitary representations of $G_1$, the representation $\nu^k \rho \times \chi_3 = \nu^k \rho \times (\nu^{-1} \tau \times \tau \times \nu \tau)$ has a unique $M_1$ model.

**Proof.** By reciprocity,
\[ \text{Hom}_{G_4}(\text{Ind}_{T_2}^G \nu^k \rho \otimes \chi_3, \text{Ind}_{G_2}^G \psi_2 \otimes 1 \otimes 1), \]
\[ \cong \text{Hom}_{T_2}(\text{Ind}_{T_2}^G \nu^k \rho \otimes \chi_3, \text{Ind}_{G_2}^G \psi_2 \otimes \text{Ind}_{G_2}^G). \]

Let $\beta = \{1, 3\}, \gamma = \{2, 2\}$. In the notation of Theorem 3.3.
\[ W^{\beta, \gamma} = \{ w_0 = id, w_1 = (1, 3, 2) \}. \]

For $w_0 = id, \beta' = \beta \cap w_0^{-1}(\gamma) = \{1, 1, 2\}; \gamma' = \gamma \cap w_0(\beta) = \{1, 1, 2\}.$
\[ F_{w_0} = \text{Ind}_{T_2}^G \nu^k \rho \otimes \nu^{-1} \tau \cong \nu^k (\nu^{-1} \tau \times \tau \times \nu \tau), \]
\[ = (\text{Ind}_{T_2}^G \nu^k \rho \otimes \nu^{-1} \tau) \otimes (\nu \tau). \]

Because the representation $\text{Ind}_{T_2}^G \nu^k \rho \otimes \nu^{-1} \tau$ has a unique Whittaker model and $(\tau \times \nu \tau) = \nu^k (\nu^{-1} \tau \times \nu \tau)$ has a unique $M_{2,1}$ model, $\nu^k \times \chi_3$ has at least one $M_1$ model.

For $w_1 = (1, 3, 2), \beta' = \beta \cap w_{1}^{-1}(\gamma) = \{1, 2, 1\}; \gamma' = \gamma \cap w_{1}(\beta) = \{2, 1, 1\}.$
\[ F_{w_1} = \text{Ind}_{T_2}^G \nu^k \rho \otimes (\nu^{-1} \tau \times \tau \times \nu \tau) \]
\[
= (\nu^{-1} \tau \times \tau) \otimes (\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu \tau).
\]

The representation \(\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu \tau\) is irreducible and has a Whittaker model. Therefore \(F_{w_1}\) has no \(\mathcal{M}_0 \otimes \mathcal{M}_{2,1}\) model, and \(\nu^k \times \chi_3\) has a unique \(\mathcal{M}_1\) model.

\[\square\]

Lemma 5.4. For \(k \neq t \pm 1\), and \(k, t \neq \pm \frac{3}{2}; \rho, \delta, \tau\) unitary representations of \(G_1\), the representation \(\nu^k \rho \times \nu^t \delta \times \chi_2 = \nu^k \rho \times \nu^t \delta \times (\nu^{-\frac{1}{2}} \tau \times \nu^{\frac{3}{2}} \tau)\) has a unique \(\mathcal{M}_1\) model.

Proof. By reciprocity,

\[
\text{Hom}_1(\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^t \delta \times \chi_2, \text{Ind}_{U_2 \times \mathcal{S}_N, \psi} \otimes 1 \otimes 1)
\]

\[
\simeq \text{Hom}_{2,2}(\text{Ind}_{1,2}^2 \nu^k \rho \times \nu^t \delta \times \chi_2, \text{Ind}_{U_2 \psi} \otimes \text{Ind}_{\mathcal{S}_N, \psi}^2 1)
\]

Let \(\beta = \{1, 1, 2\}, \gamma = \{2, 2\}\). Then

\[
W^{\beta, \gamma} = \{w_0 = \text{id}, w_1 = (2, 3), w_2 = (1, 3)(2, 4), w_3 = (1, 3, 2)\}.
\]

and the quotient

\[
F_{w_0} = (\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^t \delta) \otimes \chi_2.
\]

Since \(\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^t \delta\) is irreducible and has a unique Whittaker model and \(\chi_2\) has a unique \(\mathcal{M}_{2,1}\) model, \(\nu^k \rho \times \chi_3\) has at least one \(\mathcal{M}_1\) model. Note that

\[
F_{w_1} = (\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^{-\frac{1}{2}} \tau) \otimes (\text{Ind}_{1,1}^2 \nu^t \delta \otimes \nu^{\frac{3}{2}} \tau).
\]

and \(\text{Ind}_{1,1}^2 \nu^t \delta \otimes \nu^{\frac{3}{2}} \tau\) is irreducible and has a Whittaker model. Therefore \(F_{w_1}\) has no \(\mathcal{M}_0 \otimes \mathcal{M}_{2,1}\) model.

Since

\[
F_{w_2} = \chi_2 \otimes (\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^t \delta).
\]

and \(\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^t \delta\) is irreducible and has a Whittaker model, \(F_{w_2}\) has no \(\mathcal{M}_0 \otimes \mathcal{M}_{2,1}\) model.

Also

\[
F_{w_3} = (\text{Ind}_{1,1}^2 \nu^t \delta \otimes \nu^{-\frac{1}{2}} \tau) \otimes (\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^{\frac{3}{2}} \tau).
\]

and \((\text{Ind}_{1,1}^2 \nu^k \rho \otimes \nu^{\frac{3}{2}} \tau)\) is irreducible and has a Whittaker model. Therefore \(F_{w_3}\) has no \(\mathcal{M}_0 \otimes \mathcal{M}_{2,1}\) model, and \(\nu^k \rho \times \nu^t \delta \times \chi_2\) has a unique \(\mathcal{M}_1\) model.

\[\square\]

Lemma 5.5. If \(\chi\) is a character of \(G_n\), \(n \in \mathbb{N}\), then \(\chi\) has a unique \(\mathcal{M}_{\frac{n-1}{2}}\) model.

Proof. Two cases:
1. n=2k: Since $\text{SL}_{2k}$ is the commutator subgroup of $G_{2k}$ and characters are trivial on the commutator subgroup, $\chi$ admits an embedding in $\text{Ind}^{G_{2k}}_{\text{SL}_{2k}} 1$ and also in
$$\mathcal{M}_k = \text{Ind}^{G_{2k}}_{\text{Sp}_{2k}} 1 = \text{Ind}^{G_{2k}}_{\text{SL}_{2k}} \text{Ind}^{\text{SL}_{2k}}_{\text{Sp}_{2k}} 1.$$

2. n=2k+1: Similarly $\chi$ admits an embedding in $\text{Ind}^{G_{2k+1}}_{\text{SL}_{2k+1}} 1$, hence an embedding in
$$\mathcal{M}_k = \text{Ind}^{G_{2k+1}}_{U_1 \times \text{Sp}_{2k} \times N_1} 1 = \text{Ind}^{G_{2k+1}}_{\text{SL}_{2k+1}} \text{Ind}^{\text{SL}_{2k+1}}_{U_1 \times \text{Sp}_{2k} \times N_1} 1.$$

The embedding is unique since $\chi$ is one-dimensional. \qed

**Lemma 5.6.** In $G_n = G_{m+2k}$, if $(\pi, V_\pi)$ has a unique $\mathcal{M}_k$ model, then so does $\pi' = \nu^t \otimes \pi$, $t \in \mathbb{R}$.

**Proof.** The existence of $\mathcal{M}_k$ model for $(\pi, V_\pi)$ means that for $x \in V_\pi$, there exists a function $f_x : G \rightarrow V$, such that
1. $f_x(ug) = \psi_m(u)f_x(g)$, for $u \in U_1 \times \text{Sp}_{2k} \times N_k, g \in G_n$,
2. $f_{ax+by} = af_x + bf_y$, for $a, b \in \mathbb{C}, x, y \in G_n$,
3. $\pi(s)f_x = f_{\pi(s)x}$. That is $f_x(gs) = f_{\pi(s)x}(g)$, for $g, s \in G_n, x \in V$.
4. $f_x$ is locally constant. (That is, there exists open compact subgroup $K_{f_x} \subset G_n$ such that $f_x(gk) = f_x(g)$, for $k \in K_{f_x}, g \in G_n$.)

Let $W = \{h_x|h_x(g) = f_{\nu^t(g)x}(g), \forall x \in V, g \in G_n\}$. Then $W$ is a $\mathcal{M}_k$ model of $\pi'$, because of the verification of the following facts:
1. $h_x(ug) = f_{\nu^t(ug)x}(ug) = f_{\nu^t(g)x}(ug) = \psi_m(u)f_{\nu^t(g)x}(g)$
   $$= \psi_m(u)h_x(g).$$
2. $h_{ax+by}(g) = f_{\nu^t(ax+by)x}(g) = af_{\nu^t(g)x} + bf_{\nu^t(g)y}(g) = ah_x(g) + bh_y(g)$.
3. $\pi'(s)h_x(g) = h_x(gs) = f_{\nu^t(gs)x}(gs) = f_{\pi(s)\nu^t(gs)x}(g)$
   $$= f_{\nu^t(g)\nu^t(s)x}(g) = h_{\pi'(s)x}(g).$$
4. Because $\nu : G_n \rightarrow \mathbb{R}_{>0}$ is a homomorphism, $\nu(K) = 1$ for all compact subgroup $K$ of $G_n$. Given any $f_x$, there exists an open compact subgroup $K_{f_x}$ in $G_n$ such that $f_x(gk) = f_x(g)$, for $k \in K_{f_x}$. Then $h_x(gk) = f_{\nu^t(gk)x}(gk) = f_{\nu^t(gk)}f_x(gk) = \nu^t(g)f_x(gk) = \nu^t(gk)f_x(g)$,
   $$= f_{\nu^t(g)x}(gk) = h_x(gk).$$

By the above construction, if $\pi'$ admits two different models $\mathcal{M}_k$, $\mathcal{M}_{\nu'}$, then so does $\pi = \nu^{-1} \otimes \pi'$. This shows the uniqueness. \qed
Let $\delta_i, \rho_i, \tau_i$ be square integrable representations of $G_i$; $\chi_i$ characters of $G_i$ (we omit the subscript $i$ if $i = 1$), and $\alpha, \lambda \in \left(0, \frac{1}{2}\right)$ be real numbers.

**Theorem 5.7.** Any unitary representation on $G_5$ has one of the following expressions and induced models:

1. $\delta_5$, a square integrable representation of $G_5$, has a unique Whittaker model.

2. $\nu(\delta_5) = L(\nu^2 \delta \times \nu \delta \times \delta \times \nu^{-1} \delta \times \nu^{-2} \delta)$, a character of $G_5$ has a unique $M_2$ model.

3. Unitary representation induced from $P_{1,4}$:
   
   (a) $\delta \times \delta_1$ has a unique Whittaker model, since $\delta$ and $\delta_1$ both have Whittaker models.
   
   (b) $\delta \times \chi_4$ has a unique $M_2$ model.
   
   (c) $\delta \times L(\nu^2 \delta_2 \times \nu^{-1} \delta_2)$ has a unique $M_2$ model.

4. Unitary representation induced from $P_{2,3}$:
   
   (a) $\delta_2 \times \delta_3$ has a unique Whittaker model, since $\delta_2$ and $\delta_3$ both have Whittaker models.
   
   (b) $\delta_3 \times \chi_2$ has a unique $M_1$ model.
   
   (c) $\delta_2 \times \chi_3$ has a unique $M_1$ model.
   
   (d) $\chi_2 \times \chi_3$ has a unique $M_2$ model.

5. Unitary representation induced from $P_{1,1,3}$:
   
   (a) $\delta \times \tau \times \delta_2$ has a unique Whittaker model, since $\delta$, $\tau$ and $\delta_2$ all have Whittaker models.
   
   (b) $\delta \times \tau \times \chi_3$ has a unique $M_1$ model.
   
   (c) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \delta_3$ has a unique Whittaker model, since $\delta$ and $\delta_3$ both have Whittaker models.
   
   (d) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \chi_3$, has a unique $M_1$ model.

6. Unitary representation induced from $P_{1,1,1,2}$:
   
   (a) $\delta \times \rho \times \tau \times \delta_2$ has a unique Whittaker model, since $\delta$, $\rho$, $\tau$ and $\delta_2$ all have Whittaker models.
   
   (b) $\delta \times \tau \times \rho \times \chi_2$ has a unique $M_1$ model.
   
   (c) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \tau \times \delta_2$ has a unique Whittaker model, since $\delta$, $\tau$ and $\delta_2$ all have Whittaker models.
   
   (d) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \tau \times \chi_2$ has a unique $M_1$ model.

7. Unitary representation induced from $P_{1,2,2}$:
   
   (a) $\delta \times \delta_2 \times \delta_2$ has a unique Whittaker model, since $\delta$, $\delta_2$, and $\delta_2$ all have Whittaker models.
   
   (b) $\delta \times \delta_2 \times \chi_2$ has a unique $M_1$ model.
   
   (c) $\delta \times \chi_2 \times \chi_2$ has a unique $M_2$ model.
(d) $\delta \times \nu^\alpha \Delta_2 \times \nu^{-\alpha} \Delta_2$ has a unique Whittaker model, since $\delta, \Delta_2$ both have Whittaker models.

(e) $\delta \times \nu^\alpha \chi_2 \times \nu^{-\alpha} \chi_2$ has a unique $\mathcal{M}_2$ model.

8. Unitary representation induced from $\mathbf{P}_{1,1,1,1}$:

(a) $\delta \times \tau \times \rho \times \delta' \times \tau'$ has a unique Whittaker model, since $\delta, \tau, \rho, \delta', \tau'$ all have Whittaker models.

(b) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \tau \times \rho \times \rho'$ has a unique Whittaker model, since $\delta, \tau, \rho$ and $\rho'$ all have Whittaker models.

(c) $\nu^\alpha \delta \times \nu^{-\alpha} \delta \times \nu^\lambda \tau \times \nu^{-\lambda} \tau \times \rho$ has a unique Whittaker model, since $\delta, \tau$ and $\rho$ all have Whittaker models.

Proof. (b) Set $\chi_4 = (\nu^{-\frac{3}{2}} \rho \times \nu^{-\frac{3}{2}} \rho \times \nu^2 \rho \times \nu^2 \rho)$, where $\rho$ is an unitary representation of $G_1$. By reciprocity,

$$\text{Hom}_5(\delta \times \chi_4, \text{Ind}_U^G \psi_1 \otimes 1 \otimes 1)$$

$$\simeq \text{Hom}_{14}((\mathbf{F}_4 \otimes \text{Ind}_U^G \chi_4, \text{Ind}_U^G \psi_1 \otimes \text{Ind}_U^G 1)).$$

For $\beta = \{1,4\}, \gamma = \{1,4\}$, $W^{\beta, \gamma} = \{w_0 = id, w_1 = (1,2)\}$, and the quotient $F_{w_0} = \delta \otimes \chi_4$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_{4,2}$ model. $F_{w_1}$ has no $\mathcal{M}_0 \otimes \mathcal{M}_{4,2}$ model, because

$$F_{w_1} = \nu^{-\frac{3}{2}} \rho \otimes \text{Ind}_U^G (\nu^{-\frac{3}{2}} \rho \times \nu^2 \rho, \nu^{-\frac{3}{2}} \rho \times \nu^2 \rho)$$

and the representation $\text{Ind}_U^G \nu^{-\frac{3}{2}} \rho \otimes (\nu^{-\frac{3}{2}} \rho \times \nu^2 \rho)$ has a $\mathcal{M}_{4,1}$ model (refer to Lemma 5.3). Hence $\delta \times \chi_4$ has a unique $\mathcal{M}_2$ model.

3. (c) By reciprocity

$$\text{Hom}_5(\delta \times L(\nu^\lambda \Delta_2 \times \nu^{-\lambda} \Delta_2), \text{Ind}_U^G \psi_1 \otimes 1 \otimes 1)$$

$$\simeq \text{Hom}_{14}((\mathbf{F}_4 \otimes \text{Ind}_U^G \chi_4 \otimes \text{Ind}_U^G \psi_1 \otimes \text{Ind}_U^G 1)).$$

For $\beta = \gamma = \{1,4\}$, $W^{\beta, \beta} = \{w_0 = id, w_1 = (1,2)\}$, and the quotient $F_{w_0} = \delta \otimes L(\nu^\lambda \Delta_2 \times \nu^{-\lambda} \Delta_2)$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model by [HR] Theorem 11.1. $F_{w_0}$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. For

$$F_{w_1} = \text{Ind}_U^G \psi_1 \circ \mathbf{F}_4 \otimes L(\nu^\lambda \Delta_2 \times \nu^{-\lambda} \Delta_2),$$

with $\delta_2$ either (i) supercuspidal or (ii) Steinberg:

(i) when $\delta_2$ is supercuspidal, $F_{w_1} = 0$ has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model.

(ii) when $\delta_2$ is Steinberg, set

$$\pi = L(\nu^\lambda \Delta_2 \times \nu^{-\lambda} \Delta_2) = L(L(\Delta_1) \times L(\Delta_2)),$$
where $\delta_2 = (\nu^{\frac{1}{2}} \rho \times \nu^{-\frac{1}{2}} \rho)$, and segments $\triangle_1 = [\rho, \nu \rho], \triangle_2 = [\nu^{-1} \rho, \rho]$. By Lemma 3.8, $L(\triangle_1) \times L(\triangle_2) = \langle \nu \rho \times \rho \rangle \times (\rho \times \nu^{-1} \rho)$ has two constitutions, $L(L(\triangle_1) \times L(\triangle_2)) = \pi$ and

$$L(\triangle_1 \cup \triangle_2, \triangle_1 \cap \triangle_2) = L(\triangle_1 \cup \triangle_2) \times L(\triangle_1 \cap \triangle_2) = L([\nu^{-1} \rho, \rho, \nu \rho], \rho).$$

First.

$$\tilde{r}_{1,3}^4 \text{Ind}_{2,2}^4 (\nu \rho \times \rho) \times (\rho \times \nu^{-1} \rho),$$

has two constitutions $F_{w_0'}$ and $F_{w_1'}$, where $F_{w_0'} = \nu \rho \otimes (\text{Ind}_{1,2}^3 (\rho \times (\rho \times \nu^{-1} \rho)))$ and $F_{w_1'} = \rho \otimes (\text{Ind}_{2,1}^3 (\nu \rho \times \rho) \otimes \nu^{-1} \rho)$ are obtained from $W^{\beta' \gamma'} = \{w_0' = id, w_1' = (1, 2, 3)\}$, with $\beta' = \{2, 2\}$ and $\gamma' = \{1, 3\}$. Next.

$$\tilde{r}_{1,3}^4 \text{Ind}_{2,1}^4 (\nu \rho \times \rho \times \nu^{-1} \rho) \otimes \rho$$

also has two constitutions $F_{w_0''}$ and $F_{w_1''}$, where

$$F_{w_0''} = \nu \rho \otimes (\text{Ind}_{2,1}^3 (\rho \times \nu^{-1} \rho) \otimes \rho)$$

and $F_{w_1''} = \rho \otimes (\nu \rho \times \rho \times \nu^{-1} \rho) \otimes \rho$ are obtained from $W^{\beta'' \gamma''} = \{w_0'' = id, w_1'' = (1, 2, 3, 4)\}$, with $\beta'' = \{3, 1\}$ and $\gamma'' = \{1, 3\}$. Since

$$\tilde{r}_{1,3}^4 L(\nu^{\frac{1}{2}} \delta_2 \times \nu^{-\frac{1}{2}} \delta_2)$$

$$= \tilde{r}_{1,3}^4 \text{Ind}_{2,2}^4 (\nu \rho \times \rho) \times (\rho \times \nu^{-1} \rho) - \tilde{r}_{1,3}^4 (\nu \rho \times \rho \times \nu^{-1} \rho) \times \rho$$

$$= \rho \otimes (\text{Ind}_{2,1}^3 (\nu \rho \times \rho) \otimes \nu^{-1} \rho) - \rho \otimes (\nu \rho \times \rho \times \nu^{-1} \rho)$$

$$= \rho \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho),$$

we have

$$F_{w_1} = \text{Ind}_{1,1,3}^{1,4} \circ w_1 \circ (\delta \otimes \rho \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho))$$

$$= \rho \otimes (\text{Ind}_{1,3}^1 \delta \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho)).$$

We claim that $\text{Ind}_{1,3}^1 \delta \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho)$ has a $M_{4,1}$ model. The quotient of

$$\tilde{r}_{2,2}^4 \text{Ind}_{1,2,1}^4 \delta \otimes (\nu \rho \times \rho) \otimes \nu^{-1} \rho$$

is $\lambda = \text{Ind}_{1,1}^4 (\delta \otimes \nu \rho) \otimes \text{Ind}_{2,1}^4 (\rho \otimes \nu^{-1} \rho).$ Because $\text{Ind}_{1,1}^4 \delta \otimes \nu \rho$ has a Whittaker model and the quotient $L((\rho \times \nu^{-1} \rho)$ of $\text{Ind}_{2,1}^4 (\rho \otimes \nu^{-1} \rho)$ has a symplectic model, $\lambda$ has a $M_{4,1}$ and so does $\text{Ind}_{1,2,1}^4 \delta \otimes (\nu \rho \times \rho) \otimes \nu^{-1} \rho$. Since $\text{Ind}_{1,2,1}^4 \delta \otimes (\nu \rho \times \rho) \otimes \nu^{-1} \rho$ consists of two irreducible constitutions $\text{Ind}_{1,3}^1 \rho \otimes (\nu \rho \times \rho \times \nu^{-1} \rho)$ (with a Whittaker model), and $\text{Ind}_{1,3}^1 \delta \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho)$, the $M_{4,1}$ model must be supported in $\text{Ind}_{1,3}^1 \delta \otimes L((\nu \rho \times \rho) \times \nu^{-1} \rho).$ This proves the claim.
By disjointness of $\mathcal{M}_{4,1}$ and $\mathcal{M}_{4,2}$, $\text{Ind}^{1}_{3,1}\delta \otimes L((\nu \rho \times \rho) \times \nu^{-1}\rho)$ has no $\mathcal{M}_{4,2}$ model and we conclude that $F_{w_3}$ has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. Hence $\delta \times L((\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2) \times \nu^{-\frac{1}{2}}\delta_2)$ has a unique $\mathcal{M}_2$ model.

4. (b) Set $\chi_2 = (\nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho)$. By reciprocity,

$$\text{Hom}_5(\delta_3 \times \chi_2, \text{Ind}_{U_3 \times Sp_2, \times N_1}^{G_3} \psi_3 \otimes 1 \otimes 1)$$

$$\simeq \text{Hom}_{3,2}(\tilde{\text{Ind}}_{3,2}^5 \chi_2, \text{Ind}_{U_3}^{G_3} \psi_3 \otimes \text{Ind}_{Sp_2}^{G_2} 1).$$

For $\beta = \gamma = \{3, 2\}$, $W^{\beta, \gamma} = \{w_0 = id, w_1 = (2, 4), (3, 5), w_2 = (3, 4)\}$, and the quotient $F_{w_0} = \delta_3 \times \chi_2$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. For $F_{w_2} = \text{Ind}_{2,2}^{3,2} \circ w_2 \circ \text{Ind}_{2,2}^{3,2} \delta_3 \otimes \chi_2$, with $\delta_3$ either (i) supercuspidal or (ii) Steinberg:

(i) when $\delta_3$ is supercuspidal, $F_{w_2}$ is 0 has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Then

$$F_{w_2} = (\text{Ind}_{1,2}^{3} \nu\tau \otimes \chi_2) \otimes (\tau \times \nu^{-1}\tau).$$

Since $\langle \tau \times \nu^{-1}\tau \rangle$ has a Whittaker model, $F_{w_2}$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

For $F_{w_2} = \text{Ind}_{2,1,1}^{3,2} \circ w_2 \circ \text{Ind}_{2,2}^{3,2} \delta_3 \otimes \chi_2$, with $\delta_3$ either supercuspidal or Steinberg:

(i) when $\delta_3$ is supercuspidal, $F_{w_0} = 0$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

(ii) when $\delta_3$ is Steinberg.

$$F_{w_2} = \text{Ind}_{2,1}^{2} (\nu\tau \times \tau) \otimes \nu^{-\frac{1}{2}} \rho) \otimes (\text{Ind}_{1,1}^{2} \nu^{-1}\tau \otimes \nu^{\frac{1}{2}} \rho).$$

The representation $\text{Ind}_{1,1}^{2} \nu^{-1}\tau \otimes \nu^{\frac{1}{2}} \rho$ has a Whittaker model, so $F_{w_2}$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Hence $\delta_3 \times \chi_2$ has a unique $\mathcal{M}_1$ model.

4. (c) Set $\chi_3 = (\nu^{-\frac{1}{2}}\rho \times \rho \times \nu\rho)$. By reciprocity,

$$\text{Hom}_5(\delta_2 \times \chi_3, \text{Ind}_{U_3 \times Sp_2, \times N_1}^{G_3} \psi_3 \otimes 1 \otimes 1)$$

$$\simeq \text{Hom}_{3,2}(\tilde{\text{Ind}}_{3,2}^5 \chi_3, \text{Ind}_{U_3}^{G_3} \psi_3 \otimes \text{Ind}_{Sp_2}^{G_2} 1).$$

For $\beta = \{2, 3\}, \gamma = \{3, 2\}$.

$$W^{\beta, \gamma} = \{w_0 = id, w_1 = (1, 4, 2, 5, 3), w_2 = (2, 4, 3)\}.$$  

The quotient $F_{w_0} = (\text{Ind}_{2,1}^{3} \delta_2 \otimes \nu^{-1}\rho) \otimes (\rho \times \nu\rho)$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model, and $F_{w_2} = \chi_3 \otimes \delta_2$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Note that $F_{w_2} = \text{Ind}_{2,2,1,1}^{3,2} \circ w_2 \circ \text{Ind}_{2,2,1,1}^{3,2} \delta_2 \otimes (\nu^{-1}\rho \times \rho \times \nu\rho)$.

(i) When $\delta_2$ is supercuspidal, $F_{w_2} = 0$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

(ii) When $\delta_2$ is Steinberg, set $\delta_2 = (\nu^{\frac{1}{2}}\tau \times \nu^{-\frac{1}{2}}\tau)$. Because

$$F_{w_2} = (\text{Ind}_{1,2}^{3} \nu^{\frac{1}{2}}\tau \otimes (\nu^{-1}\rho \times \rho)) \otimes \text{Ind}_{1,1}^{2} \nu^{-\frac{1}{2}}\tau \otimes \nu\rho$$
and \( \text{Ind}_{12}^3 \nu^{-\frac{5}{2}} \tau \otimes \nu \rho \) has a Whittaker model, \( F_{w_2} \) has no \( \mathcal{M}_0 \otimes \mathcal{M}_1 \) model. Hence \( \delta_2 \times \chi_2 \) has a unique \( \mathcal{M}_1 \) model.

4. (d) Set \( \chi_3 = (\nu^{-1} \tau \times \tau \times \nu \tau), \chi_2 = (\nu^{-\frac{5}{2}} \rho \times \nu^\frac{5}{2} \rho) \). By reciprocity,

\[
\text{Hom}_5 \left( \chi_3 \times \chi_2, \text{Ind}_{12}^3 \text{Ind}_{U_3 \times \text{Sp}_2 \times N_1}^G \psi_1 \otimes 1 \otimes 1 \right)
\]

\[
\simeq \text{Hom}_4 \left( \text{Ind}_{U_3 \times \text{Sp}_2}^G \chi_3 \times \chi_2, \text{Ind}_{U_1}^G \psi_1 \otimes \text{Ind}_{\text{Sp}_4}^G \right).
\]

For \( \beta = \{3, 2\}, \gamma = \{1, 4\}, W^\beta, \gamma = \{w_0 = \text{id}, w_1 = (1, 2, 3, 4)\}, \) and the quotient \( F_{w_0} = \nu^{-1} \tau \otimes (\text{Ind}_{U_3}^G (\nu^{-1} \tau \times \tau) \otimes (\nu^{-\frac{5}{2}} \rho \times \nu^\frac{5}{2} \rho)) \). By [HR] Proposition 11.4, \( (\nu^{-1} \tau \times \tau) \times (\nu^{-\frac{5}{2}} \rho \times \nu^\frac{5}{2} \rho) \) has a unique \( \mathcal{M}_2 \) model, so \( F_{w_0} \) has a unique \( \mathcal{M}_0 \otimes \mathcal{M}_2 \) model.

By disjointness of \( \mathcal{M}_{4,1} \) and \( \mathcal{M}_{4,2} \) and the fact that \( \text{Ind}_{3,1}^3 \chi_3 \otimes \nu^\frac{5}{2} \rho \) has a \( \mathcal{M}_{4,1} \) model,

\[
F_{w_1} = \nu^{-\frac{5}{2}} \rho \otimes (\text{Ind}_{3,1}^3 \chi_3 \otimes \nu^\frac{5}{2} \rho)
\]

has no \( \mathcal{M}_0 \otimes \mathcal{M}_2 \) model. Hence \( \chi_3 \times \chi_2 \) has a unique \( \mathcal{M}_2 \) model.

5. (b) and (d) are both in the form of \( \nu^k \delta \times \nu^s \tau \times \chi_3 \), for \( k \neq \pm 1 \), and \( k, t \neq \pm 2 \). Now want to show that \( \nu^k \delta \times \nu^t \tau \times \chi_3 \) has a unique \( \mathcal{M}_1 \) model. Set \( \chi_3 = (\nu^{-1} \rho \times \nu \rho) \). By reciprocity,

\[
\text{Hom}_5 (\nu^k \delta \times \nu^t \tau \times \chi_3, \text{Ind}_{U_3 \times \text{Sp}_2 \times N_1}^G \psi_3 \otimes 1 \otimes 1)
\]

\[
\simeq \text{Hom}_3 (\text{Ind}_{1,2}^5 \text{Ind}_{1,1,3}^5 \nu^k \delta \otimes \nu^t \tau \otimes \chi_3, \text{Ind}_{U_3}^G \psi_3 \otimes \text{Ind}_{\text{Sp}_4}^G \).
\]

For \( \beta = \{1, 1, 3\}, \gamma = \{3, 2\} \),

\[
W^\beta, \gamma = \{w_0 = \text{id}, w_1 = (1, 4, 2, 5, 3), w_2 = (2, 4, 3)\},
\]

and the quotient \( F_{w_0} = (\text{Ind}_{1,1,1}^3 \nu^k \delta \otimes \nu^t \tau \otimes \nu^{-1} \rho) \otimes (\rho \times \nu \rho) \). Since \( \text{Ind}_{1,1,1}^3 \nu^k \delta \otimes \nu^t \tau \otimes \nu^{-1} \rho \) and \( \text{Ind}_{1,1,1}^3 \nu^k \delta \otimes \nu^t \tau \) both has a unique Whittaker model by Theorem 3.10, \( F_{w_0} \) has a unique \( \mathcal{M}_0 \otimes \mathcal{M}_1 \), and \( F_{w_1} = \chi_3 \otimes (\text{Ind}_{1,1}^2 \nu^k \delta \otimes \nu \tau) \) has no \( \mathcal{M}_0 \otimes \mathcal{M}_1 \) model. Since \( \text{Ind}_{1,1}^2 \nu^k \delta \otimes (\nu^{-1} \rho \times \rho) \) is irreducible and has a \( \mathcal{M}_1 \) model,

\[
F_{w_2} = (\text{Ind}_{1,1}^3 \nu^k \delta \otimes (\nu^{-1} \rho \times \rho)) \otimes (\text{Ind}_{1,1}^2 \nu^t \tau \otimes \nu \rho)
\]

has no \( \mathcal{M}_0 \otimes \mathcal{M}_1 \) model. Therefore \( \nu^k \delta \times \nu^t \tau \times \chi_3 \) has a unique \( \mathcal{M}_1 \) model.

6. (b) and (d) are both in the form of \( \nu^k \delta \times \nu^s \tau \times \nu^t \tau' \times \chi_2 \), and none of them are linked. That is, \( k, s, t \neq \pm \frac{1}{2}, \pm \frac{3}{2} \) and the difference between any pair of them are not \( \pm 1 \). We want to show that \( \nu^k \delta \times \nu^s \tau \times \nu^t \tau' \times \chi_2 \) has a unique \( \mathcal{M}_1 \) model. Set \( \chi_2 = (\nu^{-\frac{5}{2}} \rho \times \nu^\frac{5}{2} \rho) \). By reciprocity,

\[
\text{Hom}_5 (\nu^k \delta \times \nu^s \tau \times \nu^t \tau' \times \chi_2, \text{Ind}_{U_3 \times \text{Sp}_2 \times N_1}^G \psi_3 \otimes 1 \otimes 1)
\]
\[ \simeq \text{Hom}_{3,2}(r_{3,2}^5, \text{Ind}_{1,1,1,2}^\delta \otimes \nu^\delta \tau \otimes \nu^\delta \tau \otimes \chi_2, \text{Ind}_{U_3}^G \psi_3 \otimes \text{Ind}_{Sp_4}^G 1). \]

For \( \beta = \{1, 1, 2\}, \gamma = \{3, 2\}, \)

\[ W^{3,\gamma} = \{w_0 = id, w_1 = (1, 4, 2, 5, 3), w_2 = (3, 4), \cdots \} \]

and the quotient \( F_{w_0} = (\text{Ind}_{1,1,1,1}^\delta \otimes \nu^\delta \tau \otimes \nu^\delta \tau \otimes \chi_2) \otimes \chi_2 \) has a unique \( M_0 \otimes M_1 \) model. All other \( F_{w_0} \) cannot keep \( \chi_2 \) at (4, 5)-th position.

If it does, then \( w_0^{-1}(1) \leq w_0^{-1}(2) \leq w_0^{-1}(3) \) will force \( w_1 = id. \)

Therefore we cannot find another factor with a \( M_0 \otimes M_1 \) model.

Thus \( \nu^\delta \tau \otimes \nu^\delta \tau \otimes \chi_2 \) has a unique \( M_1 \) model.

7. (b) Set \( \chi_2 = (\nu^{-\frac{1}{2}} \rho \times \nu^{-\frac{1}{2}} \rho). \) By reciprocity.

\[ \text{Hom}_5(\delta \times \delta_2 \times \chi_2, \text{Ind}_{U_3 \times Sp_4 \times N_1}^G \psi_3 \otimes 1 \otimes 1) \]

\[ \simeq \text{Hom}_{3,2}(r_{3,2}^5, \text{Ind}_{1,1,2}^\delta \otimes \delta_2 \otimes \chi_2, \text{Ind}_{U_3}^G \psi_3 \otimes \text{Ind}_{Sp_4}^G 1). \]

For \( \beta = \{1, 2, 2\}, \gamma = \{3, 2\} \),

\[ W^{3,\gamma} = \{w_0 = id, w_1 = (2, 4)(3, 5), w_2 = (3, 4), w_3 = (1, 4, 3, 2)\} \]

and the quotient \( F_{w_0} = (\text{Ind}_{1,1,2}^\delta \otimes \delta_2) \otimes \chi_2 \) has a unique \( M_0 \otimes M_1 \) model. Also

\[ F_{w_1} = (\text{Ind}_{1,1,2}^\delta \otimes \chi_2) \otimes \delta_2 \]

has no \( M_0 \otimes M_1 \) model. Note that

\[ F_{w_2} = \text{Ind}_{1,1,1,1,1}^G \circ \delta_2 \circ \text{Ind}_{1,1,1,1,1}^G \delta \otimes \delta_2 \otimes \chi_2 \]

where \( \delta_2 \) is either (i) supercuspidal or (ii) Steinberg.

(i) When \( \delta_2 \) is supercuspidal, \( F_{w_2} = 0 \) has no \( M_0 \otimes M_1 \) model.

(ii) When \( \delta_2 \) is Steinberg, set \( \delta_2 = (\nu^{-\frac{1}{2}} \tau \times \nu^{-\frac{1}{2}} \tau) \). Then

\[ F_{w_3} = (\text{Ind}_{1,1,1,1}^\delta \otimes \nu^{-\frac{1}{2}} \tau \otimes \nu^{-\frac{1}{2}} \rho \otimes (\text{Ind}_{1,1,1,1}^\delta \nu^{-\frac{1}{2}} \tau \otimes \nu^{-\frac{1}{2}} \rho) \]

has no \( M_0 \otimes M_1 \) model, neither does

\[ F_{w_3} = (\text{Ind}_{1,1,1,1}^\delta \otimes \nu^{-\frac{1}{2}} \rho \otimes (\text{Ind}_{1,1,1,1}^\delta \nu^{-\frac{1}{2}} \rho). \]

Hence \( \delta \times \delta_2 \times \chi_2 \) has a unique \( M_1 \) model.

7. (c) and (e) are both in the form of \( \delta \times \nu^\alpha \chi_2 \times \nu^\alpha \chi_2 \), where \( \alpha \neq \lambda \pm 1; \alpha, \lambda \neq \pm \frac{1}{2}, \pm \frac{3}{2} \). Now we want to show that \( \delta \times \nu^\alpha \chi_2 \times \nu^\alpha \chi_2 \) has a unique \( M_2 \) model. By reciprocity, \( \text{Hom}_5(\delta \times \nu^\alpha \chi_2 \times \nu^\alpha \chi_2, \text{Ind}_{U_3 \times Sp_4 \times N_1}^G \psi_1 \otimes 1 \otimes 1) \)

\[ \simeq \text{Hom}_{1,4}(r_{1,4}^5, \text{Ind}_{1,2,2}^\delta \otimes \nu^\alpha \chi_2 \otimes \nu^\alpha \chi_2, \text{Ind}_{U_3}^G \psi_1 \otimes \text{Ind}_{Sp_4}^G 1). \]

For \( \beta = \{1, 2, 2\}, \gamma = \{1, 4\} \)

\[ W^{3,\gamma} = \{w_0 = id, w_1 = (1, 2), w_2 = (1, 2, 3, 4)\}. \]
and the quotient $F_{w_0} = \delta \otimes (\text{Ind}^1_{1,2} \nu^{\delta} \chi_2 \otimes \nu^\lambda \chi'_2)$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. Let $\chi_2 = (\nu^{-\frac{1}{2}} \tau \times \nu^{\frac{1}{2}} \tau)$ and $\chi'_2 = (\nu^{-\frac{1}{2}} \rho \times \nu^{\frac{1}{2}} \rho)$. Then

$$F_{w_1} = \nu^{-\frac{1}{2} \alpha} \tau \otimes (\text{Ind}^1_{1,1,2} \delta \otimes \nu^{\frac{1}{2} + \alpha} \tau \otimes \nu^\lambda \chi'_2)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model, because $\text{Ind}^1_{1,1,2} \delta \otimes \nu^{\frac{1}{2} + \alpha} \tau \otimes \nu^\lambda \chi'_2$ has a $\mathcal{M}_{4,1}$ model by Lemma 5.4. Also

$$F_{w_2} = \nu^{-\frac{1}{2} + \lambda} \rho \otimes (\text{Ind}^1_{1,2,1} \delta \otimes \nu^{\alpha} \chi_2 \otimes \nu^{\frac{1}{2} + \lambda} \rho)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model, since $\text{Ind}^1_{1,2,1} \delta \otimes \chi_2 \otimes \nu^{\frac{1}{2} + \lambda} \rho$ has a $\mathcal{M}_{4,1}$ model. Hence $\delta \times \nu^{\alpha} \chi_2 \times \nu^\lambda \chi'_2$ has a unique $\mathcal{M}_2$ model. \hfill \Box

We then have the following table of models of unitary representation on $G_i$, where $\alpha, \lambda \in (0, \frac{1}{2})$ are real numbers, $\delta_i; \rho_i; \tau_i$ are square-integrable representations of $G_i$ and $\chi_i$ are characters of $G_i$.

<table>
<thead>
<tr>
<th>Representation</th>
<th>Model</th>
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<tbody>
<tr>
<td>$\delta_5$</td>
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<tr>
<td>$L(\nu^\delta \delta \times \nu^\delta \delta \times \nu^{-\delta} \delta \times \nu^{-2 \delta})$</td>
<td>$\mathcal{M}_2$</td>
</tr>
<tr>
<td>$\delta \times \delta_4$</td>
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</tr>
<tr>
<td>$\delta \times \chi_4$</td>
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</tr>
<tr>
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<tr>
<td>$\chi_2 \times \chi_3$</td>
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</tr>
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<tr>
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<tr>
<td>$\nu^\delta \delta \times \nu^{-\delta} \delta \times \nu^{\lambda} \tau \times \rho$</td>
<td>$\mathcal{M}_0$</td>
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</tbody>
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References


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