Uniqueness of Shalika Models

Chufeng Nien*

Abstract

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathcal{F}$ a p-adic field and D is a quaternion division algebra over $\mathcal{F}$. This paper proves uniqueness of Shalika models for $\mathrm{GL}_{2n}(\mathbb{F}_q)$ and $\mathrm{GL}_{2n}(D)$, and re-obtain uniqueness of Shalika models for $\mathrm{GL}_{2n}(\mathcal{F})$, [1] for any $n \in \mathbb{N}$.

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Contents

1 Introduction

Let $\mathbb{F}_q$ denote a finite field of q elements, and $\mathcal{F}$ denote a p-adic field. Let F be one of the above fields, and $D = D_{\mathcal{F}}$ a quaternion division algebra over

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[1]Uniqueness of Shalika models for GL_{2n} over p-adic fields was first proved in 'Uniqueness of linear periods', by H. Jacquet and S. Rallis via the verification of multiplicity freeness of linear models and the fact: existence of Shalika models of GL_{2n} implying existence of linear models .

$\mathcal{F}$. Denote by Mat_n be the space of n-by-n matrices over F. Through out, ψ_0 denotes a nontrivial, complex, additive character of F.

Given D a quaternion division algebra over $\mathcal{F}$, then there exists a basis $\{1, i, j, k\}$ for D and the multiplication table is given by

	1	i	j	k
1	1	i	j	k
i	i	α	k	αj
j	j	$-k$	β	$-\beta i$
k	k	$-\alpha j$	βi	$-\alpha\beta$

for some suitable $\alpha, \beta \in \mathcal{F}^*$.

For $z = a+bi+cj+dk \in D$ with $a, b, c, d \in \mathcal{F}$, define the **conjugation** of z by $\bar{z} = a-bi-cj-dk$. Note that $\overline{z_1 \cdot z_2} = \bar{z}_2 \cdot \bar{z}_1$, i.e. it is an anti-involution on D (viewed as a multiplicative group). (*An* **anti-involution** τ *of a group* G *is an operator on* G *so that* $(gh)^\tau = h^\tau g^\tau, g, h \in G$ *and* $\tau^2 = id.$ *) Also we define two maps:* **reduced norm** N *and* **reduced trace** Tr *on* D *by* $Nz = z\bar{z},\ Trz = z + \bar{z}$. *There is an* **embedding** $\iota : D \hookrightarrow \mathrm{GL}(2, K)$ *defined by*

$$z = a + bi + cj + dk = (a + bi) + (c + di)j$$
$$= z_1 + z_2 j \mapsto \begin{pmatrix} z_1 & z_2\beta \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix},$$

where $K = \mathcal{F}(\sqrt{\alpha})$. *Then* $Trz = tr(\iota z), z \in D$, *where "tr" is the trace map on matrices. Under this embedding,* D *is a closed subgroup of* $\mathrm{GL}(2, K)$. *This embedding can be naturally extended to*

$$\iota : \mathrm{GL}(n, D) \hookrightarrow \mathrm{GL}(2n, K).$$

Let A *be either the field* F *(one of the* $\mathbb{F}_q$, $\mathcal{F}$*) or the quaternion division algebra* D. *In* $\mathrm{GL}_{2n}(A)$, *denote*

$$d(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix},\ g \in \mathrm{GL}_n \text{ and } u(X) = \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix},\ X \in \mathrm{Mat}_n.$$

Let $\mathrm{M}_n = \{d(g) | g \in \mathrm{GL}_n\}, \mathrm{U}_n = \{u(X) | X \in \mathrm{M}_n\}$; *and* $\mathrm{S}_n = \mathrm{M}_n\mathrm{U}_n$. **A Shalika character** *on* S_n *is given by*

$$\psi_n(d(g)u(X)) = \begin{cases} \psi_0(trX), \text{ for } s \in \mathrm{S}_n(\mathrm{F}) \\ \psi_0(tr(\iota X)), \text{ for } s \in \mathrm{S}_n(D) \end{cases}.$$

By abusing notation, we abbreviate $\psi_0(tr(\iota X))$ *by* $\psi_0(trX)$ *for* $X \in \mathrm{Mat}_n(D)$ *since no confusion should occur. Moreover, we will always refer to smooth representations when we talk about representations of groups other than finite groups.*

Let ρ *be an irreducible representation of* $\mathrm{GL}_{2n}(A)$.

Definition 1.1. *A linear functional* $\Lambda_\rho : V_\rho \mapsto \mathbb{C}$ *is called a* **Shalika functional** *of* V_ρ *if it satisfies*

$$\Lambda_\rho(\rho(s)v) = \psi_n(s)\Lambda_\rho(v) \text{ for all } s \in \mathrm{S}_n \text{ and } v \in V_\rho.$$

We say that V_ρ *has a* **Shalika model** *if there exists a nontrivial Shalika functional* Λ_ρ *satisfying the above equation. This definition is equivalent to*

$$\dim \mathrm{Hom}_{\mathrm{GL}_{2n}}(\rho, \mathrm{Ind}_{\mathrm{S}_n}^{\mathrm{GL}_{2n}} \psi_n) \geq 1,$$

since $\mathrm{Hom}_{\mathrm{GL}_{2n}}(\rho, \mathrm{Ind}_{\mathrm{S}_n}^{\mathrm{GL}_{2n}} \psi_n) \cong \mathrm{Hom}_{\mathrm{S}_n}(\rho|_{\mathrm{S}_n}, \psi_n)$ *by reciprocity.*

Definition 1.2. *Given a representation* π *of a group* G*, we say that* π *is* **multiplicity free** *or possesses the* **uniqueness** *property if* $\dim \mathrm{Hom}_G(\rho, \pi) \leq 1$*, for any irreducible representation* ρ *of* G.

Definition 1.3. *Let* $\pi = \mathrm{Ind}_L^{\mathrm{GL}_{2n}} 1$*, where* $L = \{\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, g_i \in \mathrm{GL}_n\}$*, and* 1 *denotes the trivial representation of* $\mathrm{GL}_n \times \mathrm{GL}_n$*. We say that* V_ρ *has a* **linear model** *if there exists a nontrivial intertwining operator from* V_ρ *to* π.

For general linear groups over non-archimedean local field, uniqueness of Shalika models is proved in [JR] via the verification of the multiplicity freeness of linear models. Classification of Shalika models is not yet completely established. In [Sa], Y. Sakellaridis showed necessary and sufficient conditions for an irreducible unramified principal series admitting Shalika models. In [JiS2], D. Jiang and D. Soudry showed a certain group of representations possessing Shalika models. In [JiQ], D. Jiang and Y. J. Qin defined a generalized Shalika model for SO(4n) and found the relationship between this model and Shalika model of GL(2n). The study of Shalika models interacts intensively with other related subjects. Let π *be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$*, where* $\mathbb{A}$ *is the Adele ring of a number field* F*. Then the following statements are equivalent:*

1. π *is the image of Langlands functorial lifting from* SO_{2n+1}.

2. π *has a nonzero Shalika period.*

3. *The exterior square L-function* $L(s, \pi, \wedge^2)$ *has a simple pole at* $s = 1$.

The theory has been established over years through the work of many authors in [JaS]; [CKPSS]; [GRS]; [Ki], [JiS1] and [Ji1].

Generalizations to the case of (quaternion) division algebras have been studied by various Mathematicians. In 'Kirillov theory of $\mathrm{GL}_2(D)$*, where* D *is a division algebra over a non-Archimedean local field', D. Prasad and A. Raghuram have showed the uniqueness of Shalika model and established the self-duality of irreducible representations admitting Shalika models on* $\mathrm{GL}_2(D)$*. Moreover, some extensions of the above theory (regarding to the poles of the exterior square L-function and non-vanishing of Shalika periods) in the case of* $\mathrm{GL}_2(D)$ *is given by H. Jacquet and K. Martin in [JM]. Moreover, they stated a conjecture (Jacquet-Martin conjecture) relating the existences of Shalika models for representations of* $GL_{2n}(D)$ *and* $\mathrm{GL}_{4n}(\mathcal{F})$*, which is now a theorem of [GT] by W. T. Gan and S. Takeda in the case of* $GL_2(D)$ *and* $\mathrm{GL}_4(\mathcal{F})$*. In this paper we will show uniqueness of Shalika models for* $\mathrm{GL}_{2n}, n \in \mathbb{N}$ *in the setting of p-adic fields, finite fields and p-adic quaternion division algebras. We expect that the uniqueness of Shalika models for* $\mathrm{GL}_{2n}(D)$ *(or even* $\mathrm{GL}_{2n}(\mathbb{F}_q)$*) could be proved useful in the near future. Here we present the main theorems.*

Theorem 3.2 *For any* $n \in \mathbb{N}$*, let* $G = \mathrm{GL}_{2n}(\mathbb{F}_q)$*, where* $\mathbb{F}_q$ *is a finite field. Then*

$$\dim \mathrm{Hom}_G(\rho, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) \leq 1,$$

for any irreducible representation of G.

Theorem 3.4 *Let* $G = \mathrm{GL}_{2n}$ *over either a* p*-adic field* $\mathcal{F}$ *or a quaternion division algebra* D *over* $\mathcal{F}$*. Then*

$$\dim \mathrm{Hom}_G(\rho, \mathrm{Ind}_{\mathrm{S}_{n(A)}}^G \psi_n) \leq 1,$$

for any irreducible representation of G.

2 Common Strategy

Given a finite group G*, it is known that a representation* V_π *of* G *is multiplicity free if and only if the endomorphism algebra* $\mathrm{Hom}_G(V_\pi, V_\pi)$ *is abelian.*

Moreover, when $V_\pi = \mathrm{Ind}_H^G \rho$ *is an induced representation,* $\mathrm{Hom}_G(V_\pi, V_\pi)$ *is explicitly characterized by Mackey Theorem.*

Theorem 2.1. *(Mackey's Theorem) Let* G *be a finite group. As a vector space,* $\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \pi_1, \mathrm{Ind}_{H_2}^G \pi_2)$ *is isomorphic to the space of functions*

$$\triangle : G \mapsto \mathrm{Hom}_{\mathbb{C}}(\pi_1, \pi_2) \ satisfying$$

$$\triangle(h_2 g h_1) = \pi_2(h_2) \circ \triangle(g) \circ \pi_1(h_1), \ \ for \ h_1 \in H_1, h_2 \in H_2.$$

Given such a function $\triangle$*, the corresponding intertwining operator*

$L \in \mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \pi_1, \mathrm{Ind}_{H_2}^G \pi_2)$ *is given by* $L(f_1) = \triangle * f_1$ *for* $f_1 \in \mathrm{Ind}_{H_1}^G \pi_1$*,*

where the convolution

$$\triangle * f_1(x) = \frac{1}{|G|} \sum_{g \in G} \triangle(x g^{-1}) f_1(g).$$

Especially, when $H_1 = H_2, \pi_1 = \pi_2$*, the* $\mathrm{Aut}_G(\mathrm{Ind}_{H_1}^G \pi_1)$ *is isomorphic to* $(\mathfrak{S}, *)$ *as an algebra, where the multiplication is the convolution given by*

$$\triangle_1 * \triangle_2(g) = \sum_{x \in G} \triangle_1(g x^{-1}) \triangle_2(x), \triangle_i \in \mathfrak{S}.$$

In order to show that the endomorphism algebra is abelian, identifying an anti-involution to interchange the order of factors is a common strategy.

The analogue of this method in p*-adic case is "Gelfand-Kazhdan criterion', which is first investigated in [GK] and further investigated by [BZ1] and [Gr1].*

Let $C_c^\infty(X)$ *denote the space of smooth, compactly supported functions on a* l*-adic space* X *(in the sense of [BZ1]). Let* $\mathfrak{D}(X)$ *denote the space of linear functionals on* $C_c^\infty(X)$*. Given a* p*-adic group* G*, define actions* L_g *and* R_g *on* $G; C_c^\infty(G)$ *and* $\mathfrak{D}(G)$ *as the following:*

$$L_g \cdot x = gx; \ R_g \cdot x = x g^{-1};$$

$$(L_g \cdot f)(x) = f(g^{-1} x); \ (R_g \cdot f)(x) = f(xg);$$

$$(L_g \cdot T)(f) = T(L_{g^{-1}} \cdot f); \ (R_g \cdot T)(f) = T(R_{g^{-1}} \cdot f),$$

where $g, \ x \in G; f \in C_c^\infty(G)$ *and* $T \in \mathfrak{D}(G)$*.*

Theorem 2.2. *([Ga], Gelfand-Kazhdan Criterion) Let ψ and ψ^τ be characters of a closed unimodular subgroup H of G. Suppose that there is an anti-involution τ of G so that τ stabilizes H, $\psi(h^\tau) = \psi^\tau(h)$, and τ acts trivially on all distributions T so that*

$$T(L_h\eta) = \psi(h) \cdot T(\eta);\ T(R_h\eta) = \psi^\tau(h)^{-1} \cdot T(\eta) \text{ for } \eta \in C_c^\infty(G).$$

Then

$$\dim \mathrm{Hom}_G(\pi; \mathrm{Ind}_H^G\psi) \cdot \dim \mathrm{Hom}_H(Res_H^G\tilde{\pi}; \psi^\tau) \leq 1,$$

where π is any irreducible representation of G and $\tilde{\pi}$ its contragradient.

3 Main Theorems

Define anti-involutions on $\mathrm{GL}_n(\mathrm{F})$ and $\mathrm{GL}_n(D)$ respectively by

$$\tau_n : \mathrm{GL}_{2n}(\mathrm{F}) \mapsto \mathrm{GL}_{2n}(\mathrm{F}), g \mapsto w_{2n} g^t w_{2n}^{-1},$$

$$\tau_n : \mathrm{GL}_{2n}(D) \mapsto \mathrm{GL}_{2n}(D), g \mapsto w_{2n} \bar{g}^t w_{2n}^{-1},$$

where w_{2n} is the representative given by a permutation matrix of longest Weyl elements of GL_{2n}. When n is understood, we will abbreviate τ_n by τ. Note that τ_n stabilizes S_n and ψ_n. Throughout B_n will denote the Borel subgroup of GL_n and W_n the Weyl group of GL_n.

Lemma 3.1. *Let A be either a finite field or a p-adic field or a quaternion division algebra D over $\mathcal{F}$. For each $g \in \mathrm{GL}_{2n}(A)$, there exist $s, s' \in \mathrm{S}_n$ (depending on g) such that one of the following conditions holds:*[2]
(C. 1): $sgs'^{-1} = g$, *and* $\psi_n(ss'^{-1}) \neq 1$.
(C. 2): $sgs'^{-1} = g^\tau$, *and* $\psi_n(ss'^{-1}) = 1$.

Proof. We proceed the proof by Mathematical Induction. By Bruhat decomposition

$$\mathrm{GL}_2 = \mathrm{B}_2\mathrm{W}_2\mathrm{B}_2 = \mathrm{S}_2 D \mathrm{W}_2 D \mathrm{S}_2, \text{ where } D = \{\begin{pmatrix} a & \\ & 1 \end{pmatrix} | a \in A^*\}.$$

[2]*If g satisfies* **(C. 1)** *(resp.* **(C. 2)***), then g^τ, g^{-1} and tgt', $t, t' \in \mathrm{S}_n$ will also satisfy the same condition (with different s, s'.)*

Representatives of $S_2\backslash GL_2/S_2$ can be expressed by

$$\xi_1(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ or } \xi_2(a) = \begin{pmatrix} & a \\ 1 & \end{pmatrix}, \; a \in A^*.$$

Since

$$\xi_1(a) \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \xi_1(a)^{-1} = \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}, x \in A,$$

and ψ_0 is nontrivial, there exists $x \in A$ such that $\psi_0(ax) \neq \psi_0(x)$ for $a \neq 1$. Hence $\xi_1(a)$ satisfies **(C. 1)** for $a \neq 1$. For $a = 1$, $\xi_1(a) = id$ is τ-invariant and satisfies **(C. 2)**. In the case of $A = \mathrm{F}$, $\xi_2(a), a \in \mathrm{F}^*$ is τ-invariant. In the case of $A = D$, there exists $b \in D^*$ such that $bab^{-1} = \bar{a}$ for $a \in D^*$. (Either refer to [Ra], Lemma 3.1 or check by direct computation.) Therefore, there exists $\begin{pmatrix} b & \\ & 1 \end{pmatrix} \in \mathrm{Mat}_2$ such that

$$\begin{pmatrix} b & \\ & 1 \end{pmatrix} \xi_2(a) \begin{pmatrix} b & \\ & 1 \end{pmatrix}^{-1} = \xi_2(a)^\tau$$

and **(C. 2)** holds for $\xi_2(a)$. The conclusion is then true for $n = 1$. Now we assume that the conclusion is true for $1, \cdots, n-1$.

Let $\mathrm{P}_n,\ n \in \mathbb{N}$ denote the parabolic subgroup of GL_{2n} corresponding to the partition $\{n, n\}$, and $\mathrm{W}_{\mathrm{P}_n}$ its Weyl group. Then

$$\mathrm{P}_n\backslash \mathrm{GL}_{2n}/\mathrm{P}_n \cong \mathrm{W}_{\mathrm{P}_n}\backslash \mathrm{W}_{2n}/\mathrm{W}_{\mathrm{P}_n} \leftrightarrow \mathfrak{L}_n,$$

where

$$\mathfrak{L}_n = \{(a_{i,j}) \in \mathrm{Mat}_2(\mathbb{Z}) | 0 \leq a_{i,j} \leq n, \sum_{k=1}^{2} a_{k,j} = \sum_{k=1}^{2} a_{i,k} = n, 1 \leq i, j \leq 2\}.$$

The last bijection refers to [GaRe]. Notice that the cardinality of $\mathfrak{L}_n$ is $n+1$. Moreover, we can choose representatives of $\mathrm{W}_{\mathrm{P}_n}\backslash \mathrm{W}_{2n}/\mathrm{W}_{\mathrm{P}_n}$ to be τ-invariant, given by

$$\sigma_0 = id, \sigma_k = (1, 2n)(2, 2n-1) \cdots (k, 2n+1-k), k = 1, \cdots n.$$

That is $\sigma_k = \begin{pmatrix} & & w_k \\ & I_{2n-2k} & \\ w_k & & \end{pmatrix}$, where $w_k = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix}$ is the longest Weyl element of GL_k. For $\alpha, \beta \in \mathrm{GL}_n$, denote by $d_1(\alpha) = \begin{pmatrix} \alpha & \\ & I_n \end{pmatrix}$

and $d_2(\beta) = \begin{pmatrix} I_n & \\ & \beta \end{pmatrix}$. Then the representatives of $\mathrm{S}_n\backslash\mathrm{GL}_{2n}/\mathrm{S}_n$ can be expressed by

$$\{d_1(\alpha)\sigma_k d_2(\beta) | k = 0, \cdots, n, \alpha, \beta \in \mathrm{GL}_n\}.$$

For $k \in \{0, \cdots, n\}$, fix any

$$\gamma_k = \gamma_k(\alpha, \beta) = d_1(\alpha)\sigma_k d_2(\beta) \in \mathrm{S}_n\backslash\mathrm{GL}_{2n}/\mathrm{S}_n, \alpha, \beta \in \mathrm{GL}_n.$$

Let $u_k = u_k(X) = \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix}$, where $X = \begin{pmatrix} 0 & 0 \\ X_k & 0 \end{pmatrix}$, $X_k \in \mathrm{Mat}_{n-k}$. Then $\sigma_k^{-1} u_k \sigma_k = u_k$. Let

$$n_1 = d_1(\alpha) u_k d_1(\alpha)^{-1} \in \mathrm{U}_n \text{ and } n_2 = \gamma_k^{-1} n_1 \gamma_k = d_2(\beta)^{-1} u_k d_2(\beta) \in \mathrm{U}_n.$$

Note that

$$\psi_n(n_1) = \psi_0(tr(\alpha X)) \text{ and } \psi_n(n_2) = \psi_0(tr(X\beta)) = \psi_0(tr(\beta X)).$$

Denote $\alpha = \begin{pmatrix} * & \alpha_k \\ * & * \end{pmatrix}$, and $\beta = \begin{pmatrix} * & \beta_k \\ * & * \end{pmatrix}$, where $\alpha_k, \beta_k \in \mathrm{Mat}_{n-k}$. Then

$$\alpha X = \begin{pmatrix} \alpha_k X_k & 0 \\ * & 0 \end{pmatrix}, \text{ and } tr(\alpha X) = tr(\alpha_k X_k).$$

Similarly, $tr(\beta X) = tr(\beta_k X_k)$. If $\alpha_k \neq \beta_k$, then there exists $X \in \mathrm{Mat}_n$ such that $\psi_n(tr(\alpha X)) \neq \psi_n(tr(\beta X))$. Therefore condition **(C. 1)** holds for $\gamma_k = \gamma_k(\alpha, \beta)$ with $\alpha_k \neq \beta_k$.

Assume that $\alpha_k = \beta_k$. That is

$$\gamma_k = \begin{pmatrix} * & \alpha_k & \\ * & * & \\ & & I_n \end{pmatrix} \sigma_k \begin{pmatrix} I_n & & \\ & * & \alpha_k \\ & * & * \end{pmatrix} = \begin{pmatrix} & \alpha_k & & * \\ & * & & \\ & & * & \alpha_k \\ w_k & & & \end{pmatrix}.$$

Define an equivalence relation " $\sim$ " on $g, g' \in \mathrm{GL}_{2n}$, where $g \sim g'$ means that $sgs' = g'$ for some $s, s' \in \mathrm{S}_n$. We sometimes use subscripts $\sim_{(s,s')}$ to indicate the connecting map (s, s').

If $\mathrm{rank}(\alpha_K) \neq 0$, there exist $g, h \in \mathrm{GL}_{n-k}$ such that

$$\gamma_k \sim \begin{pmatrix} g & & & \\ & I_{n+k} & & \\ & & g & \\ & & & I_{n+k} \end{pmatrix} \gamma_k \begin{pmatrix} I_{n+k} & & & \\ & h & & \\ & & I_{n+k} & \\ & & & h \end{pmatrix}$$

$$
= \begin{pmatrix} & 0 & 1 & * & * & * \\ & \delta_{k-1} & 0 & * & * & * \\ & * & * & * & * & * \\ & & & * & 0 & 1 \\ & & & * & \delta_{k-1} & 0 \\ w_k & & & & & \end{pmatrix}, \text{ for some } \delta_{k-1} \in \mathrm{Mat}_{n-k-1}.
$$

For some suitable $m, m' \in \mathrm{M}_n, u, u' \in \mathrm{U}_n$

$$
\gamma_k \sim_{(m,m')} \begin{pmatrix} * & 0 & 1 & * & * & * \\ & \delta'_{k-1} & 0 & * & * & * \\ & * & 0 & * & * & * \\ & & & 0 & 0 & 1 \\ & & & * & \delta'_{k-1} & 0 \\ w_k & & & & & * \end{pmatrix}
$$

$$
\sim_{(u,u')} \begin{pmatrix} & 0 & 1 & & & \\ & \delta''_{k-1} & 0 & * & * & \\ & * & 0 & * & * & \\ & & & 0 & 0 & 1 \\ & & & * & \delta''_{k-1} & 0 \\ w_k & & & & & \end{pmatrix} = \tilde{\gamma}_k.
$$

Let $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \in \mathrm{GL}_{2n-2}, N_i \in \mathrm{Mat}_{n-1}$ be embedded in GL_{2n} as $\begin{pmatrix} & 1 & & \\ N_1 & & N_2 & \\ & & & 1 \\ N_3 & & N_4 & \end{pmatrix}$. By induction assumption, there exist

$$
s = \begin{pmatrix} 1 & & & \\ & q & & Y \\ & & 1 & \\ & & & q \end{pmatrix} \text{ and } s' = \begin{pmatrix} p & & Z & \\ & 1 & & \\ & & p & \\ & & & 1 \end{pmatrix},
$$

where $p, q \in \mathrm{GL}_{n-1}, Y, Z \in \mathrm{Mat}_{n-1}$ such that either **(C. 1)** or **(C. 2)** holds for $\tilde{\gamma}_k$. Note that the above embedding is consistent between (τ_{n-1}, ψ_{n-1}) and (τ_n, ψ_n).

Now it suffices to show that **(C. 1)** or **(C. 2)** holds for those γ_k in the form of

$$\gamma_k = \begin{pmatrix} & 0_{n-k} & * & * \\ & * & * & * \\ & & * & 0_{n-k} \\ w_k & & & \end{pmatrix}.$$

In this case, $k \geq n-k$ and assume $k = e + n - k$ for some non-negative integer e. Then for suitable

$$s_1 = \begin{pmatrix} v_{n-k} & & & \\ & v'_k & & \\ & & v_{n-k} & \\ & & & v'_k \end{pmatrix}, s_2 = \begin{pmatrix} r_k & & & \\ & r'_{n-k} & & \\ & & r_k & \\ & & & r'_{n-k} \end{pmatrix} \in \mathrm{M}_n$$

$u, u' \in \mathrm{U}_n$; $v'_k, r_k \in \mathrm{Mat}_k$; $v_{n-k}, r'_{n-k} \in \mathrm{Mat}_{n-k}$, we can reduce γ_k to

$$\gamma_k \sim_{(s_1,s_2)} \begin{pmatrix} & 0 & * & * \\ & I_{n-k} & * & * \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix} \sim_{(u,u')} \begin{pmatrix} & 0 & 0 & \eta_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix} = \gamma'_k,$$

for some $\lambda_k, \eta_k \in \mathrm{Mat}_k$.

Denote

$$\lambda_k = \begin{pmatrix} \lambda^1_{e,n-k} & \lambda^2_{e,e} \\ \lambda^3_{n-k,n-k} & \lambda^4_{n-k,e} \end{pmatrix}; \eta_k = \begin{pmatrix} \eta^1_{n-k,e} & \eta^2_{n-k,n-k} \\ \eta^3_{e,e} & \eta^4_{e,n-k} \end{pmatrix};$$

$$C = \begin{pmatrix} C^1_{e,n-k} & C^2_{e,e} \\ {C^3}_{n-k,n-k} & C^4_{n-k,e} \end{pmatrix} = \eta_k^{-1}; \text{ and } D = \begin{pmatrix} D^1_{n-k,e} & D^2_{n-k,n-k} \\ D^3_{e,e} & D^4_{e,n-k} \end{pmatrix} = \lambda_k^{-1},$$

where subscripts denote the sizes of matrices. Let

$$s_3 = \begin{pmatrix} I_{n-k} & & & Y^1 & & \\ g^1 & I_e & & Y^2 & & \\ g^2 & g^3 & I_{n-k} & & Y^3 & Y^4 \\ & & & I_{n-k} & & \\ & & & g^1 & I_e & \\ & & & g^2 & g^3 & I_{n-k} \end{pmatrix} \in \mathrm{S}_n,$$

and $s_4 = \gamma'_k s_3 (\gamma'_k)^{-1} =$

$$\begin{pmatrix} \eta^2 g^3 C^1 + I_{n-k} & \eta^2 g^3 C^2 & 0 & \eta^1 g^1 + \eta^2 g^2 & & \\ \eta^4 g^3 C^1 & \eta^4 g^3 C^2 + I_e & 0 & \eta^3 g^1 + \eta^4 g^2 & & \\ Y^3 C^1 + Y^4 C^3 & Y^3 C^2 + Y^4 C^4 & I_{n-k} & 0 & g^2 D^1 + g^3 D^3 & g^2 D^2 + g^3 D^4 \\ & & & I_{n-k} & 0 & 0 \\ & & & \lambda^1 Y^1 + \lambda^2 Y^2 & \lambda^2 g^1 D^1 + I_e & \lambda^2 g^1 D^2 \\ & & & \lambda^3 Y^1 + \lambda^4 Y^2 & \lambda^4 g^1 D^1 & \lambda^4 g^1 D^2 + I_{n-k} \end{pmatrix}.$$

Therefore $s_4 \in \mathrm{S}_n$ if and only if

$$\left\{ \begin{array}{c} \eta^2 g^3 C^1 = \eta^2 g^3 C^2 = \lambda^2 g^1 D^2 = \lambda^4 g^1 D^2 = 0 \\ \eta^4 g^3 C^2 = \lambda^2 g^1 D^1 \\ \lambda^1 Y^1 + \lambda^2 Y^2 = \eta^4 g^3 C^1 \\ \lambda^3 Y^1 + \lambda^4 Y^2 = Y^3 C^1 + Y^4 C^3 (= p,\ say\ ) \\ Y^3 C^2 + Y^4 C^4 = \lambda^4 g^1 D^1 \end{array} \right. . \qquad (3.1)$$

Note that the last three equations in the system are equivalent to

$$\lambda_k \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} \eta^4 g^3 C^1 \\ p \end{pmatrix}, \begin{pmatrix} Y^3, & Y^4 \end{pmatrix} \eta_k^{-1} = \begin{pmatrix} p, & \lambda^4 g^1 D^1 \end{pmatrix}. \qquad (3.2)$$

(Or $\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \lambda_k^{-1} \begin{pmatrix} \eta^4 g^3 C^1 \\ p \end{pmatrix}, \begin{pmatrix} Y^3, & Y^4 \end{pmatrix} = \begin{pmatrix} p, & \lambda^4 g^1 D^1 \end{pmatrix} \eta_k.$)

Also,

$$\psi_n(s_3) = \psi_0(tr(Y^1 + Y^4)) = \psi_0(tr(D^1 \eta^4 g^3 C^1 + D^2 p + p\eta^2 + \lambda^4 g^1 D^1 \eta^4)),$$
$$= \psi_0(tr(p(D^2 + \eta^2) + D^1 \eta^4 g^3 C^1 + \lambda^4 g^1 D^1 \eta^4))$$
$$\psi_n(s_4) = \psi_0(tr(\eta^1 g^1 + \eta^2 g^2 + g^2 D^2 + g^3 D^4)) = \psi_0(tr(g^2(D^2 + \eta^2) + \eta^1 g^1 + g^3 D^4)).$$

If we let $g^1 = g^3 = 0$ and chose $Y^i, i = 1,2,3,4$ such that

$$Y^3 C^2 + Y^4 C^4 = 0, Y^1 = D^2(Y^3 C^1 + Y^4 C^3), Y^2 = D^4(Y^3 C^1 + Y^4 C^3).$$

Then $s_4 \in \mathrm{S}_n$ and

$$\psi_n(s_3) = \psi_0(tr(p(D^2 + \eta^2))) = \psi_0(tr(Y^3 C^1 + Y^4 C^3)(D^2 + \eta^2)),$$
$$\psi_n(s_4) = \psi_0(tr(\eta^2 g^2 + g^2 D^2)) = \psi_0(tr(g^2(D^2 + \eta^2))).$$

If $\eta^2 \neq -D^2$, we can choose suitable g^2 such that $\psi_n(s_3) \neq \psi_n(s_4)$. In this case, γ_k' satisfies **(C. 1)** and so does γ_k.

Suppose that $\eta^2 = -D^2$ in γ_k and the rank of η^2 equals n' . Choose $g, h \in \mathrm{Mat}_{n-k}$ such that $g\eta^2 h = \begin{pmatrix} 0 & -I_{n'} \\ 0 & 0 \end{pmatrix}$ and let

$$m = \mathrm{diag}(g, I_e, h^{-1}), m' = \mathrm{diag}(g^{-1}, I_e, h).$$

Then γ_k'

$$\sim_{(m,m')} \begin{pmatrix} & 0 & 0 & \begin{pmatrix} * & * & 0 & -I_{n'} \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k' & & & \end{pmatrix} \sim \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ * & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k'' & & * & \end{pmatrix},$$

where $\tilde{\eta}_k = \begin{pmatrix} 0 & 0 & 0 & -I_{n'} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ I_{2k-n-n''} & 0 & 0 & 0 \end{pmatrix}$, and $n'' = n - k - n'$. For suitable $u, u' \in \mathrm{U}_n$, further reduction shows that

$$\gamma_k \sim_{(u,u')} \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\lambda}_k & & & \end{pmatrix} = \tilde{\gamma}_k.$$

Next we consider

$$\tilde{\gamma}_k^{-1} = \begin{pmatrix} & 0 & 0 & \tilde{\lambda}_k^{-1} \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\eta}_k^{-1} & & & \end{pmatrix}.$$

By the same argument and the equality of η^2 and $-D^2$, we may assume that

$$\tilde{\lambda}_k^{-1} = \begin{pmatrix} 0 & 0 & I_{n'} \\ B^1_{n-k,2k-n} & 0 & 0 \\ B^3_{2k-n,2k-n} & B^4_{2k-n,n''} & 0 \end{pmatrix},$$

where subscripts denote the sizes of matrices.

Now it suffices to show that

$$\theta_k = \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\lambda}_k & & & \end{pmatrix} \text{with } \tilde{\eta}_k = \begin{pmatrix} 0 & 0 & I_{n'} \\ B^1_{n-k,2k-n} & 0 & 0 \\ B^3_{2k-n,2k-n} & B^4_{2k-n,n''} & 0 \end{pmatrix}$$

and $\tilde{\lambda}_k = \begin{pmatrix} 0 & 0 & 0 & -I_{n'} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ I_{2k-n-n''} & 0 & 0 & 0 \end{pmatrix}$ satisfies one of the conditions **(C. 1)** or **(C. 2)**. Consider the system (3.1) for θ_k. Take

$$g^1 = \begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix}, g^3 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix}, \tag{3.3}$$

where subscripts denote the sizes of matrices. Then

$$\eta^2 g^3 = 0, g^1 D^2 = 0, \eta^4 g^3 C^2 = 0, \lambda^2 g^1 D^1 = 0$$

and $s_4 \in \mathrm{S}_n$ with some suitable Y^i satisfying equation (3.2). Unless $D^1\eta^4\lambda^4 g^1 = \eta^1 g^1$ and $g^3 C^1 D^1 \eta^4 = g^3 D^4$ for all g^1, g^3 in the form of equation (3.3), there exist $s_3, s_4 \in \mathrm{S}_n$ such that **(C. 1)** holds for θ_k. (Note that $tr\lambda^4 g^1 D^1 \eta^4 = trD^1\eta^4\lambda^4 g^1$, etc..)

Next we consider those θ_k satisfying:

$$\text{Condition (3.4)} : D^1\eta^4\lambda^4 g^1 = \eta^1 g^1 \textit{ and } g^3 C^1 D^1 \eta^4 = g^3 D^4$$

$$\text{for all } g^1, g^3 \text{ in the form of equation (3.3).}$$

Let $\eta^4 = \begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix}$; $\eta^1 = \begin{pmatrix} 0 & 0 \\ T^3 & T^4 \end{pmatrix}$; and $C^1 = \begin{pmatrix} 0 & V^1 \\ 0 & V^2 \end{pmatrix}$, where $T^2, T^3, V^1 \in \mathrm{Mat}_{n''}$. Then

$$D^1\eta^4\lambda^4 g^1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n''} \end{pmatrix} \begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix} \begin{pmatrix} I_{n''} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & T^2 r_{n'',n''} \end{pmatrix}$$

$$\eta^1 g^1 = \begin{pmatrix} 0 & 0 \\ T^3 & T^4 \end{pmatrix} \begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T^3 r_{n'',n''} \end{pmatrix};$$

$$g^3 C^1 D^1 \eta^4 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & V^1 \\ 0 & V^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n''} \end{pmatrix} \begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix} \begin{pmatrix} V^1 T^2 & 0 \\ V^2 T^2 & 0 \end{pmatrix}$$

$$g^3 D^4 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix} \begin{pmatrix} I_{n''} & 0 \\ 0 & 0 \end{pmatrix}.$$

Condition (3.4) implies that

$$T^2 = T^3, V^1 T^2 = I_{n''}, V^2 = 0.$$

That is

$$\theta_k = \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\lambda}_k & & & \end{pmatrix},$$

with $\tilde{\eta}_k = \begin{pmatrix} 0 & 0 & 0 & I_{n'} \\ T^2 & T^4 & 0 & 0 \\ * & * & T^1 & 0 \\ * & * & T^2 & 0 \end{pmatrix}$, $\tilde{\lambda}_k = \begin{pmatrix} 0 & 0 & 0 & I_{2k-n'-n''} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ -I_{n'} & 0 & 0 & 0 \end{pmatrix}$. Since $V^1T^2 = I_{n''}$ implies that $\det T^2 \neq 0$,

$$\theta_k \sim \begin{pmatrix} & & & \begin{pmatrix} 0 & 0 & 0 & I_{n'} \\ T^2 & 0 & 0 & 0 \\ & E & 0 & 0 \\ & & T^2 & 0 \end{pmatrix} \\ & I_{n-k} & & \\ & & I_{n-k} & \\ \begin{pmatrix} 0 & 0 & 0 & I_{2k-n'-n''} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ -I_{n'} & 0 & 0 & 0 \end{pmatrix} & & & \end{pmatrix},$$

for some $E \in \mathrm{GL}_{2k-n-2n''}$. In the case of fields, there exists some $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = E^t$, where E^t denotes the transpose of E. In the case of quaternion division algebras, there exists some $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = \bar{E}^t$ (cf. [Ra], Lemma 3.1). In either case, there exists $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = E^\tau$. Let

$$\zeta = \mathrm{diag}(I_{n'+n''}, g, I_{n-k+2n''}, I_{n'+n''}, g, I_{n-k+2n''}).$$

Then $\zeta\theta_k\zeta^{-1} = \theta_k^\tau$ and θ_k satisfies **(C. 2)**.

□

Theorem 3.2. *For any $n \in \mathbb{N}$, let $G = \mathrm{GL}_{2n}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field. Then*

$$\dim \mathrm{Hom}_G(\rho, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) \leq 1,$$

for any irreducible representation of G.

Proof. Let $\pi = \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n$. Lemma 3.1 implies that elements

$$\triangle : G \mapsto \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ind}_{\mathrm{S}_n}^G \psi_n, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n)$$

satisfying

$$\triangle(s'gs) = \pi(s') \circ \triangle(g) \circ \pi(s), \text{ for } s, s' \in \mathrm{S}_n$$

is τ-invariant. By Theorem 2.1 $\mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{S}_n}^G \psi_n, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n)$ is abelian and the result follows. □

Lemma 3.3. *Let G denote $\mathrm{GL}_{2n}(A)$, where A is a p-adic field $\mathcal{F}$ or a quaternion division algebra D over $\mathcal{F}$. If T is a distribution on G satisfying*

$$T(L_{h_1} \circ R_{h_2}(\eta)) = \psi_n(h_1 h_2^{-1})T(\eta) \ for \ h_1, h_2 \in \mathrm{S}_n(A), \eta \in C_c^\infty(G), \tag{3.4}$$

then T is τ-invariant.

Proof. [3] Let $\mathbb{H}$ denote $\mathrm{S}_n(A)$. We verify the assumptions of Theorem 6.10 in [BZ1]. Put $\hat{\mathbb{H}} = \mathbb{H} \times \mathbb{H}$. Let $(h_1, h_2) \in \hat{\mathbb{H}}$ act on $g \in G$ and $\eta \in C_c^\infty(G)$ by

$$(h_1, h_2) \cdot g = h_1 g h_2^{-1} \text{ and } (h_1, h_2) \cdot \eta(g) = \psi_n(h_1^{-1} h_2)\eta(h_1^{-1} g h_2).$$

The assumptions of Theorem 6.10 in [BZ1] in this case are the following:

1. *The action of $\hat{\mathbb{H}}$ is constructible (same as constructive in the sense of [BZ1]), which means that the set of $\{(g, \hat{h} \cdot g) | g \in G, \hat{h} \in \hat{\mathbb{H}}\}$ is the union of finitely many locally closed subsets of $G \times G$.*

2. *For each $\hat{h} \in \hat{\mathbb{H}}$, there is $\hat{h}_\tau \in \hat{\mathbb{H}}$ such that $\hat{h} \cdot g^\tau = (\hat{h}_\tau \cdot g)^\tau$ for all $g \in G$.*

3. *$\tau^2 = \mathrm{id}$.*

4. *If T is a nonzero $\hat{\mathbb{H}}$-invariant distribution on an $\hat{\mathbb{H}}$-orbit Y, then $Y^\tau = Y$ and $T^\tau = T$.*

The conclusion is that any $\hat{\mathbb{H}}$-invariant distribution on G is also τ-invariant.

By Theorem A in 6.15 of [BZ1] the action of $\hat{\mathbb{H}}$ is constructable on $\mathrm{GL}_{2n}(\mathcal{F})$. Also $\iota\hat{\mathbb{H}}$ is constructable on $\mathrm{GL}_{4n}(K), K = \mathcal{F}(\sqrt{\alpha})$ and its closed subgroup $\mathrm{GL}_{2n}(D)$, where ι is the embedding defined earlier. The condition 1 is then verified. For condition 2, take $\hat{h}_\tau = (h_2^{-\tau}, h_1^{-\tau})$ for $\hat{h} = (h_1, h_2) \in \hat{\mathbb{H}}$. Since $\mathbb{H}$ is τ-invariant and $(\hat{h}_\tau)_\tau = \hat{h}$, τ induces an anti-involution on $\hat{\mathbb{H}}$ (still denoted by τ)

$$\tau : \hat{\mathbb{H}} \mapsto \hat{\mathbb{H}} \text{ by } \hat{h} \mapsto \hat{h}_\tau.$$

The action of $\hat{h} \in \hat{\mathbb{H}}$ satisfies that $\hat{h} \cdot g^\tau = (\hat{h}_\tau \cdot g)^\tau$ for all $g \in G$. Condition 3 is obvious. To verify condition 4, let T be a nonzero $\hat{\mathbb{H}}$-invariant distribution on an $\hat{\mathbb{H}}$-orbit $Y = \mathbb{H} g \mathbb{H}$, i.e.

$$T(\hat{h} \cdot (\eta)) = T(\eta) \text{ for all } \hat{h} = (h_1, h_2) \in \hat{\mathbb{H}} \text{ and } \eta \in C_c^\infty(Y).$$

[3]This proof mimics Theorem 2.3 [So]. We keep it here for the sake of completion.

Denote by $\hat{\mathbb{H}}_g$ the stabilizer of g in $\hat{\mathbb{H}}$. Then $Y \cong \hat{\mathbb{H}}/\hat{\mathbb{H}}_g$. Define a character $\hat{\psi}_n$ of $\hat{\mathbb{H}}$ by:

$$\hat{\psi}_n(\hat{h}) = \psi_n(h_1 h_2^{-1}) \text{ for } \hat{h} = (h_1, h_2) \in \hat{\mathbb{H}},$$

then $\hat{\psi}_n$ is τ-invariant and $C_c^\infty(Y) \cong \mathrm{ind}_{\hat{\mathbb{H}}_g}^{\hat{\mathbb{H}}} 1$ (un-normalized compact induction). We have that

$$T \in \mathrm{Hom}_{\hat{\mathbb{H}}}(\mathrm{ind}_{\hat{\mathbb{H}}_g}^{\hat{\mathbb{H}}} 1, \hat{\psi}_n) \cong \mathrm{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, Res_{\hat{\mathbb{H}}_g}\hat{\psi}_n)$$

by Frobenius reciprocity, where $\delta_{\hat{\mathbb{H}}}, \delta_{\hat{\mathbb{H}}_g}$ are the modular functions of $\hat{\mathbb{H}}$ and $\hat{\mathbb{H}}_g$ respectively. Since $|\hat{\psi}_n| \equiv 1$ and $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}$ is positive, by Schur's lemma we have

$$\text{either } \dim \mathrm{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, Res_{\hat{\mathbb{H}}_g}\hat{\psi}_n) = 0 \text{ or } \delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} = Res_{\hat{\mathbb{H}}_g}\hat{\psi}_n \equiv 1.$$

Therefore we conclude that: $\mathrm{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, Res_{\hat{\mathbb{H}}_g}\hat{\psi}_n) = 0$ for those g satisfy **(C. 1)**, i.e. there is no nontrivial $\hat{\mathbb{H}}$-invariant distribution T on such Y.

Now we consider those g satisfies **(C. 2)**, i.e.

$$\hat{k} \cdot g = g^\tau \text{ for some } \hat{k} = (k_1, k_2) \in \hat{\mathbb{H}} \text{ with } \hat{\psi}_n(\hat{k}) = 1.$$

We may assume that $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} \equiv 1$, since otherwise $\hat{\mathbb{H}}$-invariant distribution on such Y is trivial. Note that $\hat{k} \cdot g = g^\tau$ implies the double coset $Y = \mathbb{H}g\mathbb{H}$ is τ-invariant. It remains to show that $T^\tau = T$. In our case T is proportional (see 6.12 of [BZ1]) to

$$T_1(\eta) = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h} \cdot g)\hat{\psi}_n^{-1}(\hat{h})d\hat{h},$$

where $d\hat{h}$ is a left $\hat{\mathbb{H}}$-invariant measure on $\hat{\mathbb{H}}/\hat{\mathbb{H}}_g$. We have

$$T_1^\tau(\eta) = T_1(\eta^\tau) = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta((\hat{h} \cdot g)^\tau)\hat{\psi}_n^{-1}(\hat{h})d\hat{h}$$

$$= \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}_\tau \cdot g^\tau)\hat{\psi}_n^{-1}(\hat{h})d\hat{h}$$

$$= \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}_\tau \cdot \hat{k} \cdot g)\hat{\psi}_n^{-1}(\hat{h})d\hat{h} = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}' \cdot g)\hat{\psi}_n^{-1}(\hat{h}')\hat{\psi}_n(\hat{k})d\hat{h}'.$$

Last equality is obtained by change of variables $\hat{h}' = \hat{h}_\tau \cdot \hat{k}$ along with our assumption that $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} \equiv 1$ and the fact that $\hat{\psi}_n$ is τ-invariant. Since $\hat{\psi}_n(\hat{k}) = 1$, we have

$$T_1^\tau(\eta) = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}' \cdot g)\hat{\psi}_n^{-1}(\hat{h}')d\hat{h}' = T_1(\eta).$$

□

Theorem 3.4. *Let $G = \mathrm{GL}_{2n}(A)$, where A is either a p-adic field $\mathcal{F}$ or a quaternion division algebra D over $\mathcal{F}$. Then $\dim \mathrm{Hom}_G(\rho, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) \leq 1$, for any irreducible representation ρ of G.*

Proof. We have obtained

$$\dim \mathrm{Hom}_G(\pi; \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) \cdot \dim \mathrm{Hom}_{\mathrm{S}_n}(Res_{\mathrm{S}_n}^G \tilde{\pi}; \psi_n) \leq 1,$$

for any irreducible representation π of G from the previous theorem and the Gelfand-Kazhdan criterion. It suffices to show that if π has a nontrivial Shalika functional then $\tilde{\pi}$ will also have one. Assume that Λ_π is a nontrivial Shalika functional for π, i.e.

$$\Lambda_\pi(\pi(h)v) = \psi_n(h)\Lambda_\pi(v) \text{ for all } h \in \mathrm{S}_n \text{ and } v \in V_\pi.$$

Define a representation π' on the same vector space V_π by

$$\pi'(g)v = \pi(\xi g^{-\tau}\xi^{-1})v,$$

where $\xi = \mathrm{diag}(I_n, -I_n)$. Then $\psi(\xi s^{-\tau}\xi^{-1}) = \psi(s)$ for $s \in \mathrm{S}_n$, and Λ_π is also a Shalika functional for π'.

In the case of $\mathcal{F}$, define another representation π'' on the same vector space V_π by

$$\pi''(g)v = \pi(g^{-t})v.$$

Then $\pi'' \cong \tilde{\pi}$ by Theorem 7.3 in [BZ1]. Since $\xi g^{-\tau}\xi^{-1}$ is conjugate to g^{-t} , we have $\pi' \cong \tilde{\pi}$.

In the case of D, define another representation π'' on the same vector space V_π by

$$\pi''(g)v = \pi(\eta\bar{g}^{-t}\eta^{-1})v,$$

where $\eta(i,j) = (-1)^i \delta_{i,2n-j+1}$. Then $\pi'' \sim \tilde{\pi}$ by Theorem 3.1 [Ra]. Since $\xi g^{-\tau} \xi^{-1}$ is conjugate to $\eta \bar{g}^{-t} \eta^{-1}$, we have $\pi' \sim \pi'' \sim \tilde{\pi}$.

In either case,

$$\dim \mathrm{Hom}_G(\tilde{\pi}; \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) = \dim \mathrm{Hom}_G(\pi'; \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n)$$

$$= \dim \mathrm{Hom}_{\mathrm{S}_n}(\pi'|_{\mathrm{S}_n}; \psi_n) \geq 1,$$

which completes the proof. □

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Chufeng Nien, nienpig@mail.ncku.edu.tw, Department of Mathematics, National Cheng Kung University, Tainan 701, TAIWAN