Homework 10

Propertres of Invertible Transformations / Matrices.

1. No. If A is a $m\times n$ matrix with $m\neq n$,

we can consider the linear transformation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

$$T(v) = Av \quad \forall v \in \mathbb{R}^n.$$

For the standard basis β and γ for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, we have $[T]_\beta^\gamma = A$.

Thus A is invertible $\Leftrightarrow$ T is an isomorphism.

Hence it remains to show that T is not an isomorphism.

$\because m\neq n$, either $m>n$ or $m<n$.

If $m>n$, then T is not onto, and

if $m<n$, then T is not one-to-one.

(Hw 8 #4)

$\therefore T$ is not an isomorphism. $\therefore A$ is not invertible.

2\. If $T^2 = T \circ T = 0$ (Consider $T: V \longrightarrow V$ with $V \neq \{0\}$)

Suppose that T is invertible. Then $\exists\, S: V \longrightarrow V$

s.t. $S \circ T = T \circ S = id_V$. (the identity map, $id_V(v) = v \ \forall v$)

Then $T = id_V \circ T = S \circ T \circ T = S \circ T^2$

$= S \circ 0 = 0$

$\Rightarrow$ T is not onto except $V = \{0\}$. $\times$

Computations of Inverses.

1\. (a)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{E^{-1\cdot 1+2}\,\cdot} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \xrightarrow{E^{-1\cdot 2}\,\cdot} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{E^{-2\cdot 2+1}\,\cdot} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right).$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

(b)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{E^{-1\cdot 1+3}\,\cdot} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E^{-1\cdot 2+1}\,\cdot} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{E^{-1\cdot 3+2}\,\cdot} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E^{1\cdot 3+1}\,\cdot} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \qquad \therefore \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

(c)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{E^{-2\cdot 1+3}\cdot E^{-1\cdot 1+2}\,\cdot} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{(interchange)}]{E^{2,4}\,\cdot} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \cdots$$

(不會算的請儘快找老師或助教)

Ans: $\begin{pmatrix} \frac{-2}{5} & \frac{1}{5} & \frac{3}{5} & 0 \\ -1 & -1 & 1 & -1 \\ \frac{4}{5} & \frac{-2}{5} & \frac{-1}{5} & 0 \\ \frac{3}{5} & \frac{1}{5} & \frac{-2}{5} & 0 \end{pmatrix}$

2. (a)

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix} \xrightarrow{E^{1,2}\cdot} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 2 & 4 & -5 \end{pmatrix} \xrightarrow{E^{-2\cdot 1+3}\cdot} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 2 & -3 \end{pmatrix}$$

$$\xrightarrow{E^{-\frac{1}{2}\cdot 2}\cdot} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{pmatrix} \xrightarrow{E^{-2\cdot 2+3}\cdot} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E^{2\cdot 3+2}\cdot} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E^{1\cdot 3+1}\cdot} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E^{-1\cdot 2+1}\cdot} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \operatorname{rank} \begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix} = 3.$$

(b)

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix} \xrightarrow{\cdots} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \operatorname{rank} = 4.$$

3. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{E^{-1\cdot 1+2}\cdot} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{E^{-1\cdot 1+3}\cdot} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$\xrightarrow{E^{(-\frac{1}{2})\cdot 2}\cdot} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{E^{1\cdot 2+3}\cdot} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E^{-1\cdot 3+1}\cdot} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow{E^{-2\cdot 2+1}\cdot} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\therefore \left[E^{-2\cdot 2+1}\right]\left[E^{-1\cdot 3+1}\right]\left[E^{1\cdot 2+3}\right]\left[E^{-\frac{1}{2}\cdot 2}\right]\left[E^{-1\cdot 1+3}\right]\left[E^{-1\cdot 1+2}\right]\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \left[E^{-1\cdot 1+2}\right]^{-1}\left[E^{-1\cdot 1+3}\right]^{-1}\left[E^{-\frac{1}{2}\cdot 2}\right]^{-1}\left[E^{1\cdot 2+3}\right]^{-1}\cdot$$

$$\left[E^{-1\cdot 3+1}\right]^{-1}\cdot\left[E^{-2\cdot 2+1}\right]$$

$$= \left[E^{1\cdot 1+2}\right]\left[E^{1\cdot 1+3}\right]\left[E^{-2\cdot 2}\right]\left[E^{-1\cdot 2+3}\right]\left[E^{1\cdot 3+1}\right]\left[E^{2\cdot 2+1}\right].$$