Homework 11.

Rank Computations / Row Reduced Echelon Forms

1. (a) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \xrightarrow{E^{-2\cdot 1+2}\ \cdot} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{\cdot E^{-2\cdot 1+2} \cdot E^{-1\cdot 1+3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ rank $= 1$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{E^{-1\cdot 1+4} \cdot E^{-1\cdot 1+2}} \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix}$

$\xrightarrow{E^{1\cdot 2+4} \cdot E^{-1\cdot 2+3} \cdot E^{-1\cdot 2+1}} \begin{pmatrix} 1 & 0 & 6 & 1 & 0 \\ 0 & -2 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -1 & 0 \end{pmatrix}$

$\xrightarrow{E^{34}\ \cdot} \begin{pmatrix} 1 & 0 & 6 & 1 & 0 \\ 0 & -2 & -3 & 0 & 1 \\ 0 & 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\cdot E^{\frac{1}{2}\cdot 2+5} \cdot E^{\frac{1}{4}\cdot 2+4}]{E^{-\frac{1}{2}\cdot 3+2} \cdot E^{1\cdot 3+1} \cdot}$

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ rank $= 3$

2. $\begin{pmatrix} 1 & 2 & a \\ 1 & 1 & 2a \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{E^{-1\cdot 1+2}\,\cdot} \begin{pmatrix} 1 & 2 & a \\ 0 & -1 & a \\ 0 & 1 & 3 \end{pmatrix}$

$\xrightarrow{E^{1\cdot 2+3}\cdot E^{2\cdot 2+1}\,\cdot} \begin{pmatrix} 1 & 0 & 3a \\ 0 & -1 & a \\ 0 & 0 & a+3 \end{pmatrix}$

$\xrightarrow{\cdot E^{-3a\cdot 1+3}\cdot E^{a\cdot 2+3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & a+3 \end{pmatrix}$

$\therefore$ rank $= 2 \iff a+3=0 \iff a=-3$

rank $= 3 \iff a \neq -3$.

Since either $a=-3$ or $a \neq 3$,

the matrix is either rank 2 or rank 3.

$\therefore$ "rank $= 1$" is impossible.

3. Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x) = Ax$.

$S: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. $S(x) = (cA)x$.

By definition of rank, we have that

$\operatorname{rank} A = \operatorname{rank} T$ and $\operatorname{rank}(cA) = \operatorname{rank} S$.

Note that

$$x \in N(T) \Leftrightarrow T(x) = 0 \Leftrightarrow Ax = 0$$

$$\Leftrightarrow cAx = 0 \Leftrightarrow S(x) = 0 \Leftrightarrow x \in N(S)$$

$\therefore N(T) = N(S) \Rightarrow \operatorname{nullity}(T) = \operatorname{nullity}(S)$

$\therefore \operatorname{rank} A = \operatorname{rank} T = n - \operatorname{nullity}(T)$

$= n - \operatorname{nullity}(S) = \operatorname{rank} S = \operatorname{rank}(cA)$.

4. Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x) = Ax$.

Let $\beta = \{e_1, \cdots, e_n\}$, $\gamma = \{f_1, \cdots, f_m\}$ be the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$, resp.

$\text{rank } A = m \Rightarrow \text{rank } T = m$

$\therefore$ We have that T is onto.

Then $\exists\, v_1, \cdots, v_m \in \mathbb{R}^n$ s.t. $T(v_i) = f_i$.

Then we define the linear transformation

$$S: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

by taking $S(f_i) = v_i$, $1 \le i \le m$.

Then $T \circ S(f_i) = T(v_i) = f_i$

$\Rightarrow T \circ S = id_{\mathbb{R}^m}$

$\Rightarrow I_m = [id_{\mathbb{R}^m}]_\gamma^\gamma = [T \circ S]_\gamma^\gamma$

$= [T]_\beta^\gamma [S]_\gamma^\beta = A \cdot [S]_\gamma^\beta$.

Hence we have that $B = [S]_\gamma^\beta$ satifies that $AB = I_m$.

Note that the matrix B is not necessary unique!

5. $\begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix} \xrightarrow{E^{12}\cdot} \begin{pmatrix} 1 & 1 & 1 & 1 & -3 \\ 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix}$

$\xrightarrow{E^{-2\cdot 1+4}\cdot E^{-1\cdot 1+3}\cdot E^{-2\cdot 1+2}\cdot} \begin{pmatrix} 1 & 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$

$\xrightarrow{E^{-1\cdot 3+1}\cdot E^{-2\cdot 3+2}\cdot E^{-1\cdot 3+4}\cdot} \begin{pmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{E^{-1\cdot 2+1}} \underbrace{\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$

↑ row echelon form of A

nullity $A = 5 - 3 = 2$.