Row Reduced Echelon Form

1. Determine, with sufficient reasons, if the following matrices are in row reduced echelon form:

(a) $$\begin{pmatrix} 1 & 2 & 1 & 4 & -9 & 0 & 5 & 7 & 4 \\ 0 & 0 & 0 & 1 & 4 & 0 & 3 & 8 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Systems of Linear Equations

1. Describe the set of solutions of the following systems of linear equations using Gaussian eliminations.

(a) $$\begin{cases} x_1 - 2x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = 6 \\ 3x_1 - 5x_2 = 7 \\ x_1 + 5x_3 = 9 \end{cases}$$

(b) $$\begin{cases} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6 \end{cases}$$

2. Solve the following system of linear equations, given that the coefficient matrix is invertible:

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_1 + x_2 + x_3 = 6 \\ 2x_1 - 2x_2 + x_3 = 4 \end{cases}$$

3. Prove that if the coefficient matrix $A \in Mat_{m\times n}$ has rank m, then the set of solutions is non-empty. That is, there is at least one solution to the system.

Invertibility and Determinants

1. Are the following linear transformations invertible?

(a) $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 3x - 2y \end{pmatrix}$

(b) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+2y+2z \\ 3x+2y+2z \\ x+3y-2z \end{pmatrix}$

2. Prove that for an upper triangular matrix, that is, $A = (a_{ij}) \in Mat_{n\times n}$ with $a_{ij} = 0 \; \forall \; i > j$, we have

$$detA = \Pi_{i=1}^{n} a_{ii} = a_{11}a_{22}\cdots a_{nn}.$$

3. Finish the proof mentioned in class: for any $A, B \in Mat_{n\times n}, \det(AB) = \det A \cdot \det B$.

4. Using previous two problems, compute the determinant of the following matrix. A computation straight from definition will only yield very little points.

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 4 \\ 2 & -1 & 0 & 1 & 7 \\ 3 & 0 & 1 & 2 & 8 \\ 2 & 2 & 3 & 3 & 5 \\ 1 & -2 & 3 & 9 & 5 \end{pmatrix}$$

5. Prove that rotation in $\mathbb{R}^2$ preserves area. That is, if R_θ is the rotation transformation by θ, the parallelogram spanned by u, v is equal to the parallelogram spanned by $R_\theta u$ and $R_\theta v$.