Homework 12.

Row Reduced Echelon form.

1\. (a) No. $\because j_4 = j_5$

The first nonzero entry in the forth row is $a_{47} \Rightarrow j_4 = 7$.

The first nonzero entry in the fifth row is $a_{57} \Rightarrow j_5 = 7$.

But a row reduced echelon form provides for $j_4 < j_5$.

(b) Yes, since $j_1 = 1 < j_2 = 2 < j_3 = 3$

and $a_{1j_1} = a_{2j_2} = a_{3j_3} = 1$.

Systems of Linear Equations.

1. $$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & -3 & 1 & 6 \\ 3 & -5 & 0 & 7 \\ 1 & 0 & 5 & 9 \end{pmatrix} \xrightarrow{E^{-1\cdot 1+4} \cdot E^{-3\cdot 1+3} \cdot E^{-2\cdot 1+2} \cdot} \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 6 & 8 \end{pmatrix}$$

$$\xrightarrow{E^{-2\cdot 2+4} \cdot E^{-1\cdot 2+3} \cdot E^{2\cdot 2+1} \cdot} \begin{pmatrix} 1 & 0 & 5 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The set of solutions is

$$\left\{ (x_1, x_2, x_3) \,\middle|\, \begin{array}{l} x_1 + 5x_3 = 9 \\ x_2 + 3x_3 = 4 \end{array} \right\} \quad (\text{let } x_3 = t)$$

$$= \left\{ (-5t+9, -3t+4, t) \,\middle|\, t \in \mathbb{R} \right\}$$

(b)

$$\begin{pmatrix} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{pmatrix}$$

$$\xrightarrow{E^{12}\cdot} \begin{pmatrix} 1 & -1 & 1 & 2 & -1 & 2 \\ 2 & -2 & -1 & 6 & -2 & 1 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{pmatrix}$$

$$\xrightarrow{E^{-2\cdot 1+2}\cdot E^{-4\cdot 1+3}\cdot} \begin{pmatrix} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 3 & 3 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{E^{1\cdot 2+3}\cdot E^{\frac{1}{3}\cdot 2+1}} \begin{pmatrix} 1 & -1 & 0 & \frac{4}{3} & -1 & 1 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 0 & 5 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{E^{\frac{-4}{15}\cdot 3+1}\cdot E^{\frac{-2}{5}\cdot 3+2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \frac{-19}{15} & \frac{19}{15} \\ 0 & 0 & -3 & 0 & \frac{-2}{5} & \frac{-13}{5} \\ 0 & 0 & 0 & 5 & 1 & -1 \end{pmatrix} \quad \left(\begin{array}{l} \text{let} \\ x_2 = s \\ x_5 = t \end{array} \right)$$

$\therefore$ The set of solutions is

$$\left\{ (x_1, x_2, x_3, x_4, x_5) \;\middle|\; \begin{array}{l} x_1 - x_2 - \frac{19}{15}x_5 = \frac{19}{15} \\ -3x_3 - \frac{2}{5}x_5 = \frac{-13}{5} \\ 5x_4 + x_5 = -1 \end{array} \right\}$$

$$= \left\{ \left(s + \frac{19}{15}t + \frac{19}{15},\ s,\ \frac{-2}{15}t + \frac{13}{15},\ \frac{-1}{5}t - \frac{1}{5},\ t \right) \;\middle|\; s, t \in \mathbb{R} \right\}.$$

2. Method I (use inverse matrix):

Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 47/17 \\ 27/17 \\ 28/17 \end{pmatrix}$$

※考試時要寫下求反矩陣的過程，不可以直接寫答案！

Method II (Cramer's rule)

$$\Delta = \det\begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} = (1+4+6) - (-6+2-2) = 17$$

$$\Delta x_1 = \det\begin{pmatrix} 1 & 2 & -3 \\ 6 & 1 & 1 \\ 4 & -2 & 1 \end{pmatrix} = 47$$

$$\Delta x_2 = \det\begin{pmatrix} 1 & 1 & -3 \\ 1 & 6 & 1 \\ 2 & 4 & 1 \end{pmatrix} = 27$$

$$\Delta x_3 = \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 6 \\ 2 & -2 & 4 \end{pmatrix} = 28$$

$$\therefore\ x_1 = \frac{\Delta x_1}{\Delta} = \frac{47}{17},\quad x_2 = \frac{\Delta x_2}{\Delta} = \frac{27}{17},\quad x_3 = \frac{\Delta x_3}{\Delta} = \frac{28}{17}.$$

3. $A \in \mathrm{Mat}_{m\times n}$ has rank m,

$\Rightarrow$ the linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $Tx = Ax$,

is onto.

$\therefore \forall y \in \mathbb{R}^m$, $\exists x \in \mathbb{R}^n$ s.t. $T(x) = y$.

$\therefore \forall y \in \mathbb{R}^m$, $\exists x \in \mathbb{R}^n$ s.t $Ax = y$.

$\therefore$ The system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \quad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_n \end{cases}$$

has at least one solution if

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ has rank m.

Invertibility and Determinants

1. (a) $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

$\Rightarrow [T]_\beta = \begin{pmatrix}2 & 1\\3 & -2\end{pmatrix}$ where β is the standard basis for $\mathbb{R}^2$.

Then $\det\begin{pmatrix}2 & 1\\3 & -2\end{pmatrix} = -7 \neq 0 \Rightarrow \begin{pmatrix}2 & 1\\3 & -2\end{pmatrix}$ is invertible

$\Rightarrow [T]_\beta$ is invertible $\Rightarrow T$ is invertible.

(b) $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\-2\end{pmatrix}$

$[T]_\beta = \begin{pmatrix}2 & 2 & 2\\3 & 2 & 2\\1 & 3 & -2\end{pmatrix}$ where β is the standard basis for $\mathbb{R}^3$.

Then $\det\begin{pmatrix}2 & 2 & 2\\3 & 2 & 2\\1 & 3 & -2\end{pmatrix} = 10 \neq 0 \Rightarrow [T]_\beta$ is invertible

$\Rightarrow T$ is invertible.

2. Let $A = (a_{ij}) \in Mat_{n\times n}$ be an upper triangle matrix.

Use the induction on n:

For $n=1$, $A = (a_{11})$, $\det A = a_{11} = \prod_{i=1}^{1} a_{11}$. OK.

Suppose for $n=k$, $\det A = \prod_{i=1}^{k} a_{ii} = a_{11} \cdot \cdots \cdot a_{kk}$.

is true.

If $n = k+1$, then

$$A = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1(k+1)} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right)$$

where

$$B = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2(k+1)} \\ \vdots & & & \\ a_{(k+1)2} & & \cdots & a_{(k+1)(k+1)} \end{pmatrix} \in Mat_{k\times k}$$

is upper triangle.

Thus $\det A = a_{11} \cdot \det B$

$$= a_{11} \cdot \prod_{i=2}^{k+1} a_{ii} \quad \text{(By induction hypothesis)}$$

$$= \prod_{i=1}^{k+1} a_{ii}.$$

Hence by induction, the proof is done.

3. In class, it is proved :

① If A is not invertible, then

$$\det(AB) = (\det A)(\det B) = 0.$$

② Suppose A is invertible

(i) If A is an elementary matrix, then

$$\det(AB) = (\det A)(\det B).$$

Hence it remains to show the case : A is invertible but A is not a elementary matrix.

Note that every invertible matrix can be expressed as a product of elementary matrices.

$\therefore \exists E_1, \cdots, E_n$: elementary matrices such that

$$A = E_n \cdot E_{n-1} \cdot \cdots \cdot E_1 .$$

$$\Rightarrow \det(AB) = \det\left(E_1 (E_{n-1} \cdot \cdots \cdot E_1 B\right)$$

$$= (\det E_1)(\det(E_{n-1} \cdot \cdots \cdot E_1 B))$$

$$= \cdots = (\det E_n) \cdot \cdots \cdot (\det E_1)(\det B)$$

$$= \det(E_n E_{n-1}) \cdot (\det E_{n-2}) \cdots (\det E_1)(\det B)$$

$$= \cdots = \det(E_n E_{n-1} \cdots E_1) \det B$$

$$= (\det A)(\det B).$$

4.

$$A=\begin{pmatrix} 1 & 2 & 3 & 3 & 4 \\ 2 & -1 & 0 & 1 & 7 \\ 3 & 0 & 1 & 2 & 8 \\ 2 & 2 & 3 & 3 & 5 \\ 1 & -2 & 3 & 9 & 5 \end{pmatrix} \xrightarrow{E^{-1\cdot1+5}\cdot E^{-2\cdot1+4}\cdot E^{-3\cdot1+3}\cdot E^{-2\cdot1+2}} \begin{pmatrix} 1 & 2 & 3 & 3 & 4 \\ 0 & -5 & -6 & -5 & -1 \\ 0 & -6 & -8 & -7 & -4 \\ 0 & -2 & -3 & -3 & -3 \\ 0 & -4 & 0 & 6 & 1 \end{pmatrix}$$

$$\xrightarrow[\det=-1]{\cdot E^{24}} \begin{pmatrix} 1 & 4 & 3 & 3 & 2 \\ 0 & -1 & -6 & -5 & -5 \\ 0 & -4 & -8 & -7 & -6 \\ 0 & -3 & -3 & -3 & -2 \\ 0 & 1 & 0 & 6 & -4 \end{pmatrix} \xrightarrow{E^{1\cdot2+5}\cdot E^{-3\cdot2+4}\cdot E^{-4\cdot2+3}} \begin{pmatrix} 1 & 4 & 3 & 3 & 2 \\ 0 & -1 & -6 & -5 & -5 \\ 0 & 0 & 16 & 13 & 14 \\ 0 & 0 & 15 & 12 & 13 \\ 0 & 0 & -6 & 1 & -9 \end{pmatrix}$$

$$\xrightarrow[\cdot E^{34}]{E^{35}\cdot} \begin{pmatrix} 1 & 4 & 3 & 3 & 2 \\ 0 & -1 & -6 & -5 & -5 \\ 0 & 0 & 1 & -6 & -9 \\ 0 & 0 & 12 & 15 & 13 \\ 0 & 0 & 13 & 16 & 14 \end{pmatrix} \xrightarrow{E^{-13\cdot3+5}\cdot E^{-12\cdot3+4}\cdot} \begin{pmatrix} 1 & 4 & 3 & 3 & 2 \\ 0 & -1 & -6 & -5 & -5 \\ 0 & 0 & 1 & -6 & -9 \\ 0 & 0 & 0 & 87 & 121 \\ 0 & 0 & 0 & 94 & 131 \end{pmatrix}$$

$$\xrightarrow{E^{-\frac{94}{87}\cdot4+5}\cdot} \begin{pmatrix} 1 & 4 & 3 & 3 & 2 \\ 0 & -1 & -6 & -5 & -5 \\ 0 & 0 & 1 & -6 & -9 \\ 0 & 0 & 0 & 87 & 121 \\ 0 & 0 & 0 & 0 & \frac{23}{87} \end{pmatrix}$$

Note that

$$\begin{cases} \det E^{a\cdot r+s} = 1 \\ \det E^{rs} = -1 \end{cases}$$

$$\therefore \det A = (\text{所有基本矩陣的}\det\text{相乘})\cdot\left(1\cdot(-1)\cdot(1)\cdot(87)\cdot\left(\tfrac{23}{87}\right)\right)$$

$$= (-1)^3\cdot(-23) = 23.$$

5. $R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$

$\Rightarrow [R_\theta]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where β is the standard basis for $\mathbb{R}^2$.

- The area of the parallelogram spanned by $R_\theta u$ and $R_\theta v$

$= (\det[R_\theta]_\beta) \cdot$ (the area of the parallelogram spanned by u and v)

$=$ the area of the parallelogram spanned by u and v

since $\det[R_\theta]_\beta = \det \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = 1$.