Name and Student ID: ______________________________

Homework 13, Analytic Geometry and Matrices

Change of Basis

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix},$$

with respect to $\beta = \{e_1, e_2\}$, the standard basis of $\mathbb{R}^2$. Also let

$$\beta' = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

be an alternative basis. Write down $[T]_{\beta'}^{\beta'}$.

2. If $A, B \in \mathrm{Mat}_{n\times n}$ are similar, prove that
   (a) $\operatorname{rank} A = \operatorname{rank} B$.
   (b) $\operatorname{nullity} A = \operatorname{nullity} B$.
   (c) $\det A = \det B$.

Diagonalization

1. For the following matrices, determine if they are diagonalizable. For each diagonalizable matrix A, find change of coordinate matrix Q so that $Q^{-1}AQ$ is diagonal matrix.
   (a) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$.
   (b) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$.
   (c) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$.

2. Prove, by applying induction on k, that if $\lambda_1, \ldots, \lambda_k$ are *distinct* eigenvalues of $T : V \to V$, with $v_1, \ldots, v_k$ corresponding eigenvectors, then $v_1, \ldots, v_k$ are linearly independent.

3. Prove that for a linear transformation $T : V \to V$ on an n-dimensional vector space, if the characteristic polynomial of T has n distinct real roots, then T is diagonalizable.