Homework 13

Change of basis:

1. $[T]_\beta^\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot e_1 + 1 \cdot e_2$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot e_1 + 2 \cdot e_2$

$\Rightarrow Q = [id_{\mathbb{R}^2}]_{\beta'}^{\beta} = \left(\left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\beta \; \left[T\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Then $[T]_{\beta'}^{\beta'} = Q^{-1} [T]_\beta^\beta \, Q$

$= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$.

2. A and B are similar $\Rightarrow \exists\, Q$ invertible such that $B = Q^{-1} A Q$.

(a) $\because$ invertible matrices preserve ranks

$\therefore \operatorname{rank} B = \operatorname{rank} Q^{-1} A Q = \operatorname{rank} A$.

(b) $\operatorname{nullity} A = n - \operatorname{rank} A = n - \operatorname{rank} B = \operatorname{nullity} B$.

(c) $\because \det(Q^{-1}) = \dfrac{1}{\det Q}$

$\therefore \det B = \det(Q^{-1} A Q) = (\det Q)(\det A)(\det Q^{-1}) = \det A$.

Diagonalization

1. (a) $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

characteristic polynomail :

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$

$\because$ A is a 2x2 matrix and A has two distinct eigenvalue $(-2$ and $5)$.

$\therefore$ A is diagonalizable.

$\lambda = -2$: Solve $(A+2I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = \frac{-4}{3} x_2$$ (x_2 is free)

$$\therefore E_{-2} = \mathcal{N}(A+2I) = \text{span}\left\{\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right\}$$

$\lambda = 5$: Solve $(A-5I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2$$ (x_2 is free)

$$\therefore E_5 = \mathcal{N}(A-5I) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$\Rightarrow Q = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}$. Then $Q^{-1}AQ = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$.

1. (b) $A=\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

$$f(\lambda)=\det(A-\lambda I)=\det\begin{pmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{pmatrix}$$

$$=(3-\lambda)\left((7-\lambda)(-5-\lambda)+32\right)=(3-\lambda)(\lambda^2-2\lambda-3)$$

$$=-(\lambda-3)^2(\lambda+1)$$

$\lambda=3$: Solve $(A-3I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$

$\Rightarrow \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix}$ $\quad x_1=x_2$ $\quad$ (x_2, x_3 are free)

$\Rightarrow E_3=N(A-3I)=\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\lambda=-1$: Solve $(A+\lambda)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$

$$\Rightarrow \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & \frac{-4}{3} \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore \begin{array}{l} x_1=\frac{2}{3}x_3 \\ x_2=\frac{4}{3}x_3 \end{array}$$ (x_3 is free)

$\Rightarrow E_{-1}=N(A+\lambda)=\operatorname{span}\left\{\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}\right\}$

$\therefore$ A is diagonalizable and $Q=\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$

s.t. $Q^{-1}AQ=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

1. (c) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) - 2 - 2 + (4-\lambda) - 2(1-\lambda) + 2(3-\lambda)$$

$$= -\lambda^3 + 8\lambda^2 - 20\lambda + 16 = -(\lambda-2)^2(\lambda-4)$$

$\lambda = 2$: Solve $(A-2I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow x_1 = -x_2 - x_3$, $(x_2, x_3$ are free$)$

$\therefore E_2 = N(A-2I) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = 4$: Solve $(A-4I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore x_1 = -x_3$, $x_2 = -2x_3$ $(x_3$ is free$)$

$\therefore E_4 = N(A-4I) = \text{span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$\therefore A$ is diagonalizable and $Q = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

s.t. $Q^{-1}AQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

2. If $k=1$, $\{v_1\}$ is linearly independent since $v_1 \neq 0$. $(av=0 \Rightarrow a=0)$

Suppose for $k=n$, $\{v_1, \cdots, v_n\}$ is linearly independent

Then for $k=n+1$.

$$* \quad a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + a_{n+1} v_{n+1} = 0 \quad (1)$$

Apply $T - \lambda_{n+1} id_V$, then we get that

$$a_1(\lambda_1 - \lambda_{n+1}) v_1 + a_2(\lambda_2 - \lambda_{n+1}) v_2 + \cdots + a_n(\lambda_n - \lambda_{n+1}) v_n + a_{n+1}(\lambda_{n+1} - \lambda_{n+1}) v_{n+1}$$

$$= \sum_{i=1}^{n} a_i (\lambda_i - \lambda_{n+1}) v_i = 0$$

$\because \{v_1, \cdots, v_n\}$ is linearly independent

$\therefore a_i(\lambda_i - \lambda_{n+1}) = 0 \quad \forall \ 1 \le i \le n.$

$\Rightarrow a_i = 0 \quad \forall \ 1 \le i \le n$ since $\lambda_i \neq \lambda_{n+1}$ for all $1 \le i \le n$

Then (1) reduces to $a_{n+1} v_{n+1} = 0$

$\therefore a_{n+1} = 0 \quad (\because v_{n+1} \neq 0)$

$\therefore a_1 = a_2 = \cdots = a_n = a_{n+1} = 0$

$\therefore \{v_1, v_2, \cdots, v_n, v_{n+1}\}$ is linearly independent.

3. Suppose T has n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$.

Let v_i be an eigenvector of T corresponding to λ_i.

Then $\{v_1, \cdots, v_n\}$ is linearly independent.

$\Rightarrow$ $\beta = \{v_1, \cdots, v_n\}$ forms a basis for V.

$\because$ $T(v_i) = \lambda_i v_i$, we have that

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$ is a diagonal matrix.

$\therefore$ T is diagonalizable.