Name and Student ID: ______________________________

Solutions for Homework 1 Supplementary Problems

1. What is wrong with the "set"

$$A = \{x \mid x \notin x\}?$$

Is anything of the form $\{x \mid P(x)\}$ a set? (Please google "Russell's paradox").

> **Solution:** If A is a set, any object either belongs to A or does not belong to A. In particular, either $A \in A$ or $A \notin A$. However, from the definition of A, it implies that
>
> $$A \in A \iff A \notin A,$$
>
> which is a contradiction. Therefore, not every class of the form $\{x \mid P(x)\}$ is a set. A rigorous and mathematically precise definition of "set" actually does not exist! The most commonly (and recently) accepted version of set theory is to view sets *axiomatically*, that is, define sets to be certain objects with certain properties (e.g. subset, equality, ... etc) without any justification. The school of set theory is known as Zermelo-Fraenkel set theory (with axiom of choice).

2. Consider the unit sphere

$$\mathbb{S}^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

and the "north pole" $N = (0, 0, 1) \in \mathbb{S}^2$. Consider the mapping

$$\Phi : \mathbb{S}^2 \backslash N \to \mathbb{R}^2$$

defined by

$$\Phi(x, y, z) = \frac{(x, y)}{1 - z}.$$

(a) Is Φ well defined on it domain?

> **Solution:** Yes. The only point of problem (singularity) is when $z = 1$ on $\mathbb{S}^2$, which is exactly the north pole N we removed.

(b) Place the center of $\mathbb{S}^2$ at the origin $(0, 0, 0)$. For every $(x, y, z) \in \mathbb{S}^2 \backslash N$, write down the parametric equation of the line l going through N and (x, y, z).

> **Solution:** The line l going through $(0, 0, 1)$ and (x, y, z) has parametric equation
>
> $$(x(t), y(t), z(t)) = t(x, y, z - 1).$$
>
> Note that z is not free here, but subject to the defining equation $x^2 + y^2 + z^2 = 1$ for $\mathbb{S}^2$.

(c) What is the point of intersection between l and xy plane (ie. $z = 0$)? Any relation to $\Phi(x, y, z)$?

Solution: For l to cross xy plane, $z(t) = t(1-z)$ has to be zero and therefore at $t = \frac{1}{z-1}$. Plugging into the parametric equation of l, we see that the point of intersection is

$$\left(\frac{z}{1-z}, \frac{y}{1-z}, 0\right).$$

In other words, l crosses xy plane at $(\Phi(x,y,z), 0)$.

(d) Explain, loosely using languages introduced in class and perhaps some drawing, that Φ defines a coordinate for $\mathbb{R}^2$.

Solution: $\Phi : \mathbb{S}^2\backslash N \to \mathbb{R}^2$ is one-to-one (and onto), with the inverse map $\Phi^{-1} : \mathbb{R}^2 \to \mathbb{S}^2\backslash N$ given by

$$\Phi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, 1 - \frac{2}{x^2+y^2+1},\right).$$

Readers may readily check that $\Phi^{-1}(x,y) \in \mathbb{S}^2\backslash N$.

For a pictorial explanation of one-to-one, just observe that the line through a point $(x,y,0)$ and $N \in \mathbb{S}^2$ passes through $\mathbb{S}^2\backslash N$ exactly once and therefore the mapping Φ is one-to-one. If there are two distinct points on P_1 and $P_2 \in \mathbb{S}^2\backslash N$ corresponding to one $(x,y,0)$, then P_1, P_2 and N are three distinct colinear points on a sphere which is impossible (a line can only intersect a sphere at 0, 1, or 2 points).

This is the well known "*stereographic projective coordinates*" of $\mathbb{R}^2$, globally defined for all $\mathbb{R}^2$. For 10 point extra credit, construct the stereographic projective coordinates for $\mathbb{R}^n$.

Solution: The stereographic projective coordinate for $\Phi : \mathbb{S}^n\backslash N \to \mathbb{R}^n$ is, naturally,

$$\Phi(x_1, \ldots, x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n).$$