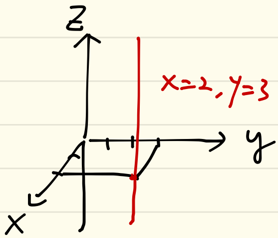
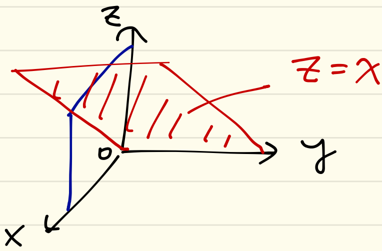
Homework 2.

Problem from Thomas' Calculus :

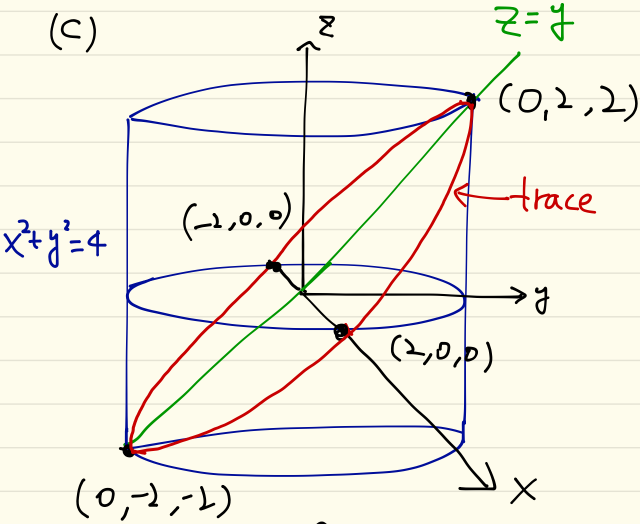
1. (a)



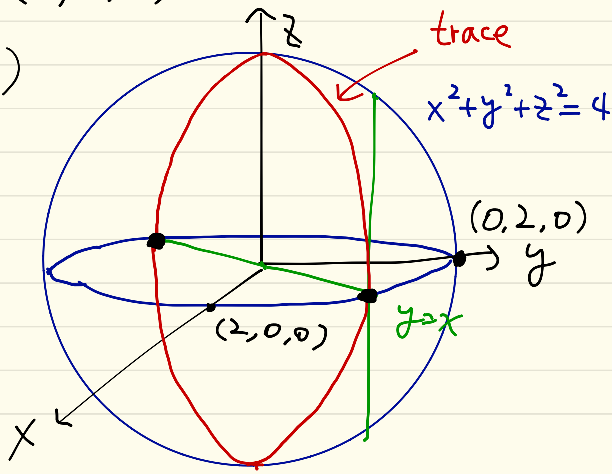
(b)



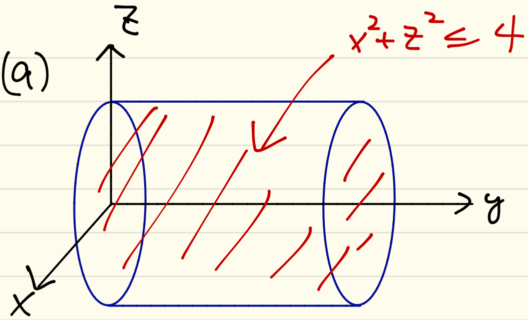
(c)



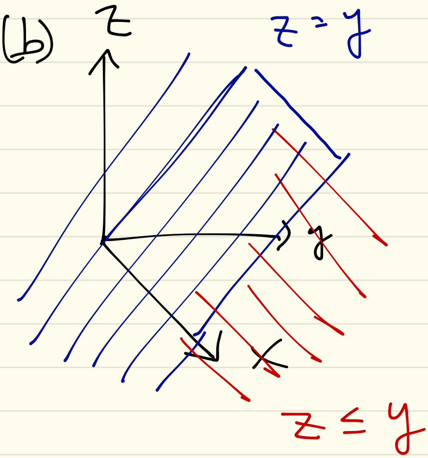
(d)



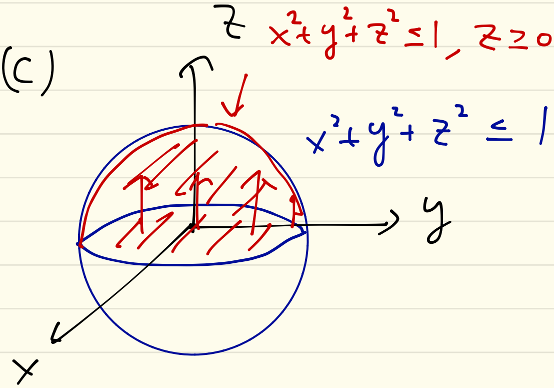
2. (a)


z
y
x
$x^2+z^2 \le 4$

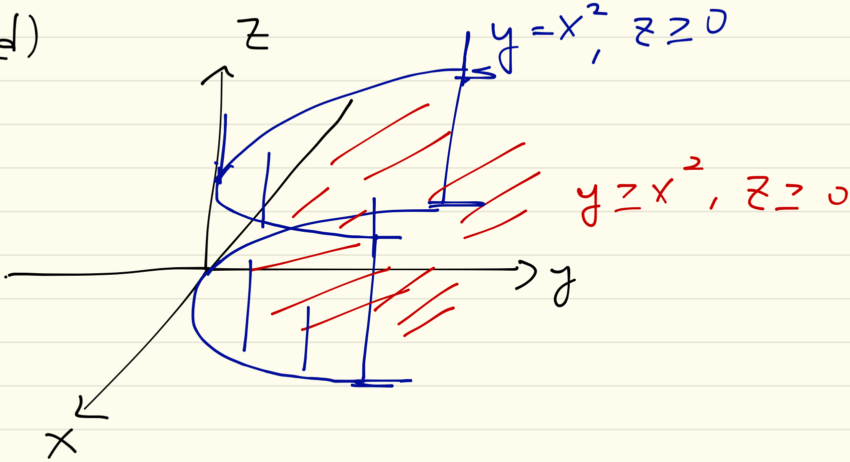
(b)


z
y
x
$z=y$
$z \le y$

(c)


z
y
x
$x^2+y^2+z^2 \le 1,\ z \ge 0$
$x^2+y^2+z^2 \le 1$

(d)


z
y
x
$y=x^2,\ z \ge 0$
$y \ge x^2,\ z \ge 0$

3. Let P be the set of all points equidistant from the point $(0,0,2)$ and the xy-plane.

Then $(x,y,z) \in P \Leftrightarrow x^2+y^2+(z-2)^2 = |z|$ $\square$

Problems from class:

1. $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$.

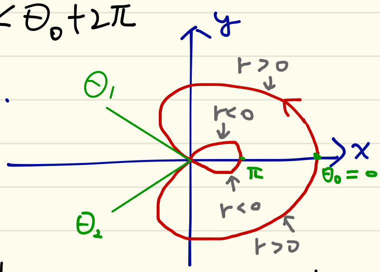
(A) For the case $\left|\frac{a}{b}\right| \in (0,1)$, that is $|a|<|b|$. we can take θ_0 such that $|r(\theta_0)| = |a|+|b|$, and consider the angle $\theta_0 \le \theta \le \theta_0 + 2\pi$.

Note that it is easy to take

$$\theta_0 < \theta_1 < \theta_0 + \pi < \theta_2 < \theta_0 + 2\pi$$

s.t. $r(\theta_1) = r(\theta_2) = 0$.

eg.



In this situation, the sign change occurs, and it produces an inner loop during the process of sign change.

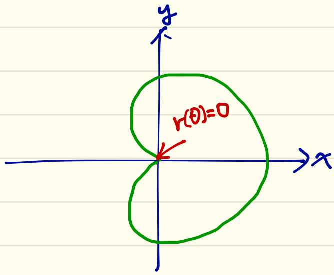
The trace of $r(\theta)$, $\theta_1 \le \theta \le \theta_2$ forms an inner loop.

(B) For the case $\left|\frac{a}{b}\right|=1$, that is $|a|=|b|$.

There is only one $\theta \in [0, 2\pi)$ s.t. $r(\theta)=0$.

In this situation, there is a cusp at the origin. Hence the geometric shape is a heart.

eg.



For (C) and (D), we have to realize how to classify a limoçon is either dimpled or convex under the assumption $\left|\frac{a}{b}\right|>1$.

Observe that in this situation ($r=a+b\cos\theta$, $a>b>0$).

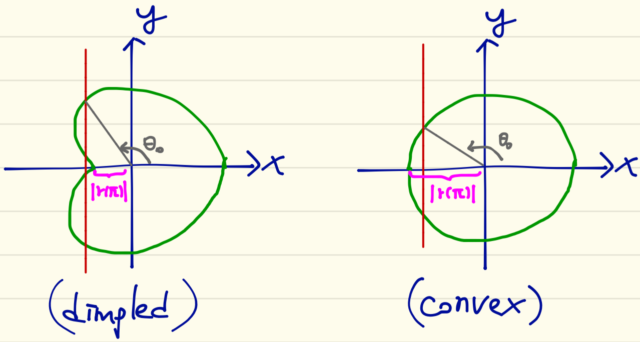
r takes the maximum value $a+b$ when $\theta=0$, and

r takes the minimum value $a-b>0$ when $\theta=\pi$.

Moreover, r is decreasing when $\theta \in (0,\pi)$ and increasing when $\theta \in (\pi, 2\pi)$.

Hence the possible dimple occurs only when θ nears to π.

For the reason, to classify such limaçon is convex or not, we just consider some vertical lines as following graphs :



Then it is convex

$\Longleftrightarrow \forall \ \frac{\pi}{2} < \theta_0 < \pi$ s.t. $|r(\theta_0)\cos\theta_0| < |r(\pi)|$

$$|r(\theta_0)\cos\theta_0| - |r(\pi)|$$
$$= -(a\cos\theta_0 + b\cos^2\theta_0) - (a-b)$$
$$= (1+\cos\theta_0)[\,b(1-\cos\theta_0) - a\,] < 0$$

$\Longleftrightarrow \quad b(1-\cos\theta_0) < a \quad \forall \ \frac{\pi}{2} < \theta_0 < \pi$

$\Longleftrightarrow \quad 2b < a \quad \Longleftrightarrow \quad \frac{a}{b} > 2$.

So (C) appears when $\frac{a}{b} \in (1,2)$

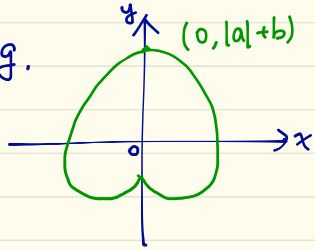
(D) appears when $\frac{a}{b} \geq 2$.

2. (i) $r = a + b\sin\theta$, $b > 0$:

The point with max $|r|$ is $(0, |a|+b)$.

orientation: $\uparrow$

eg.

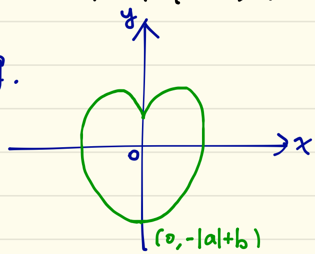


(ii) $r = a + b\sin\theta$, $b < 0$:

The point with max $|r|$ is $(0, -|a|+b)$.

orientation: $\downarrow$

eg.

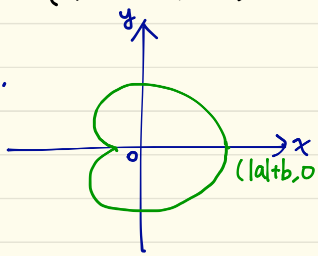


(iii) $r = a + b\cos\theta$, $b > 0$

The point with max $|r|$ is $(|a|+b, 0)$

orientation: $\longrightarrow$

eg.

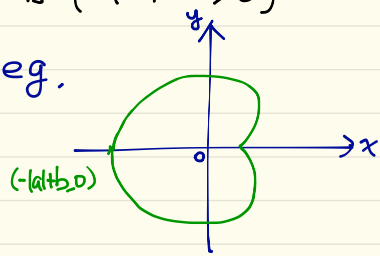


(iv) $r = a + b\cos\theta$, $b < 0$

The point with max $|r|$ is $(-|a|+b, 0)$

orientation: $\longleftarrow$

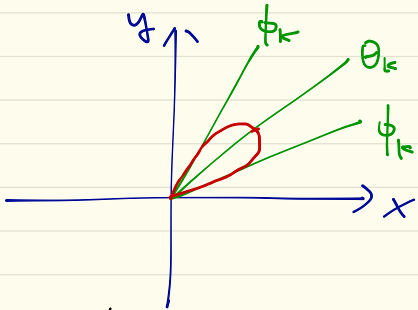
eg.



3. For the case $r = a\sin(n\theta)$, $|r(\theta_k)|$ take the maximum value $|a|$ when $\theta_k = (2k-1)\pi / 2n$, where $k = 1, 2, \cdots, 2n$.

And $r(\phi_k) = 0$ when $\phi_k = \frac{k\pi}{n}$ for $k = 0, 1, 2, \cdots, 2n$.

The trace of $r(\theta)$ forms a petal if $\phi_{k-1} \le \theta \le \phi_k$.



(the trace of $r(\theta)$, $\phi_{k-1} \le \theta \le \phi_k$ is represented by θ_k.)

Note that for $k = 1, \cdots, n$, we get n different petals (since the angle is from 0 to π) and for $k = n+1, \cdots, 2n$, we also get n different petals (the angle is from π to 2π).

Moreover, for $1 \le k \le n$, $n+1 \le j \le 2n$, the traces of θ_k and θ_j are the same only when $j = n+k$.

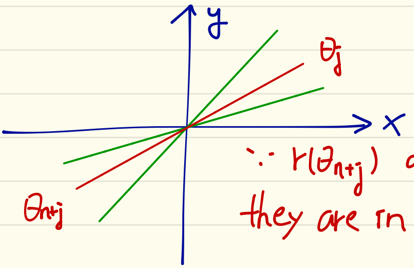
Hence the next page, we discuss how to classify they are the same or not.

When n is odd, for $1 \le j \le n$

$$r(\theta_{n+j}) = a\sin\left(\frac{(2n+2j-2)\pi}{2}\right) = a\sin(n\theta_j + n\pi)$$

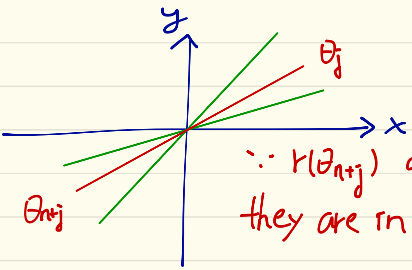
$$= -a\sin(n\theta_j) = -r(\theta_j).$$

Thus the n petals for $k = n+1, \cdots, 2n$ are the same with $k = 1, \cdots, n$. So it has n petals.



$\because$ $r(\theta_{n+j})$ and $r(\theta_j)$ have opposite sign, they are in the same region.

When n is even, for $1 \le j \le n$, $r(\theta_{n+j}) = r(\theta_j)$, so the n petals for $k = n+1, \cdots, 2n$ are different from $k = 1, \cdots, n$. So it has $2n$ petals.



$\because$ $r(\theta_{n+j})$ and $r(\theta_j)$ have the same sign they are in different regions.

For the case $r = a\cos\theta$, the proof is similar. It suffices to replace θ_k by $\dfrac{2k\pi}{2n}$.