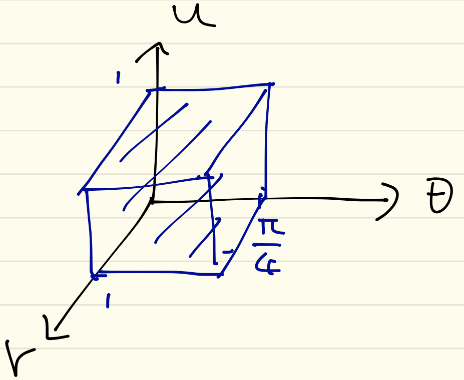
Homework 3

Problems concering coordinates :

1. $x = r\cos\theta,\ y = r\sin\theta,\ z = u$

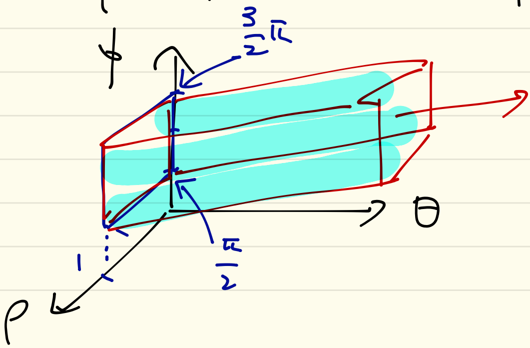
$$\begin{cases} x^2+y^2 \le 1 \\ y \ge 0 \\ y \le x \\ 0 \le z \le 1 \end{cases} \iff \begin{cases} r^2 \le 1 \\ r\sin\theta \ge 0 \\ r\sin\theta \le r\cos\theta \\ 0 \le z \le 1 \end{cases} \iff \begin{cases} 0 \le r \le 1 \\ 0 \le \sin\theta \le \cos\theta \\ 0 \le z \le 1 \end{cases}$$

$$\iff \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{4} \\ 0 \le z \le 1 \end{cases}$$



2. $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

$$\begin{cases} x^2+y^2+z^2 \le 1 \\ z \le 0 \end{cases} \iff \begin{cases} \rho^2 \le 1 \\ \rho\cos\phi \le 0 \end{cases} \iff \begin{cases} 0 \le \rho \le 1 \\ \frac{\pi}{2} \le \phi \le \frac{3}{2}\pi \end{cases}$$



Problems concering conic sections :

1. $R(x,y) = (\cos\phi\, x - \sin\phi\, y,\ \sin\phi\, x + \cos\phi\, y)$

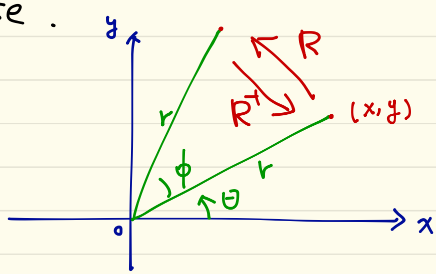
let $x = r\cos\theta$, $y = r\sin\theta$, then

$R(x,y) = R(r\cos\theta, r\sin\theta)$

$= (r\cos\theta\cos\theta - r\sin\theta\sin\phi,\ r\cos\theta\sin\phi + r\sin\theta\cos\phi)$

$= (r\cos(\theta+\phi),\ r\sin(\theta+\phi))$

is the point which is given by rotating (x,y) by the angle ϕ, counterclockwise.

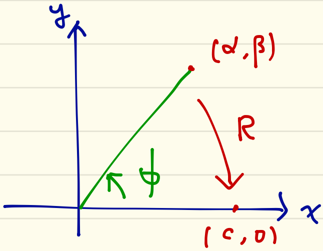


R^{-1} is the map which rotates every point on $\mathbb{R}^2$ by the angle $(-\phi)$, counterclockwise.
(i.e. rotates by ϕ clockwise)

So we have that

$$R^{-1}(x,y) = (\cos(-\phi)x - \sin(-\phi)y,\ \sin(-\phi)x + \cos(-\phi)y)$$
$$= (\cos\phi\, x + \sin\phi\, y,\ -\sin\phi\, x + \cos\phi\, y).$$

2.



First we may assume $\beta \geq 0$ and $c > 0$. Want to find a rotation S such that $S(c,0) = (\alpha, \beta)$. Then $R = S^{-1}$.

Let $0 \leq \phi \leq \pi$ with $\cos\theta = \alpha / \sqrt{\alpha^2+\beta^2}$.

Then $S(x,y) = (\cos\phi\, x - \sin\phi\, y,\ \sin\phi\, x + \cos\phi\, y)$.

$\therefore R = S^{-1}$ $\left(\cos\phi = \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}},\ \sin\phi = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}} \right.$

$$\Rightarrow R(x,y) = (\cos\phi\, x + \sin\phi\, y,\ -\sin\phi\, x + \cos\phi\, y)$$

$$= \left(\frac{\alpha x + \beta y}{\sqrt{\alpha^2+\beta^2}},\ \frac{-\beta x + \alpha y}{\sqrt{\alpha^2+\beta^2}} \right)$$

Clearly $R(\alpha,\beta) = \left(\sqrt{\alpha^2+\beta^2},\ 0\right)$,

$R(-\alpha,-\beta) = \left(-\sqrt{\alpha^2+\beta^2},\ 0\right)$.

$\therefore c = \sqrt{\alpha^2+\beta^2}$.

3. Let $c = \sqrt{\alpha^2+\beta^2}$, $b = \sqrt{a^2-c^2}$. And assume $\beta > 0$

The ellipse which we want is obtained by rotating the ellipse $\left\{ (X, Y) \in \mathbb{R}^2 \;\middle|\; \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \right\}$

by the map R that takes $\pm(c, 0)$ to $\pm(\alpha, \beta)$.

Take $R(X, Y) = \left(\frac{\alpha X - \beta Y}{\sqrt{\alpha^2+\beta^2}}, \frac{\beta X + \alpha Y}{\sqrt{\alpha^2+\beta^2}} \right)$

Let $(x, y) = R(X, Y)$

$$\Rightarrow (X, Y) = R^{-1}(x, y) = \left(\frac{\alpha x + \beta y}{\sqrt{\alpha^2+\beta^2}}, \frac{-\beta x + \alpha y}{\sqrt{\alpha^2+\beta^2}} \right)$$

So the equation $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$ turns out

$$\frac{\left(\frac{\alpha x + \beta y}{\sqrt{\alpha^2+\beta^2}}\right)^2}{a^2} + \frac{\left(\frac{-\beta x + \alpha y}{\sqrt{\alpha^2+\beta^2}}\right)^2}{b^2} = 1$$

$$\Rightarrow (\alpha^2 b^2 + \beta^2 a^2)x^2 + 2\alpha\beta(b^2 - a^2)xy + (\alpha^2 a^2 + \beta^2 b^2)y^2 = (abc)^2$$

is the equation.

4. We may assume $q > s$. Find T, α, β s.t.

$$(p,q) \xrightarrow[\text{translates}]{T} (\alpha,\beta), \quad (r,s) \xrightarrow[\text{translates}]{T} (-\alpha,-\beta)$$

$$\begin{cases} \alpha = \frac{1}{2}(p-r), \quad \beta = \frac{1}{2}(q-s) \\ T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ T(x,y) = (x-p+\alpha, \ y-q+\beta). \end{cases}$$

Now let $c = \sqrt{\alpha^2+\beta^2}$, $b = \sqrt{a^2-b^2}$.

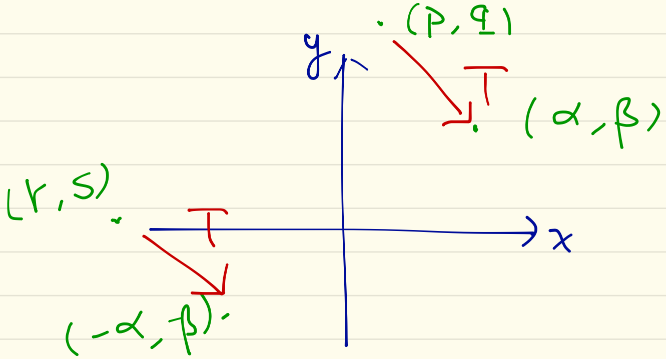
The equation of the ellipse with foci $\pm(\alpha,\beta)$ and length of major axis $2a$ is

$$(\alpha^2b^2+\beta^2a^2)X^2 + 2\alpha\beta(b^2-a^2)XY + (\alpha^2a^2+\beta^2b^2)Y^2 = (abc)^2$$

Now $X = x-p+\alpha$, $Y = y-q+\beta$

$\Rightarrow$ The equation we want is

$$(\alpha^2b^2+\beta^2a^2)(x-p+\alpha)^2 + 2\alpha\beta(b^2-a^2)(x-p+\alpha)(y-q+\beta)$$
$$+ (\alpha^2a^2+\beta^2b^2)(y-q+\beta)^2 = (abc)^2.$$



5. Want to write down the equation of a hyperbola with general foci (p,q) and (r,s) and length of major axis $2a$ for some $a \le \frac{1}{2}\sqrt{(p-r)^2+(q-s)^2}$.

Assume $q > s$.

Let $\alpha = \frac{1}{2}(p-r)$, $\beta = \frac{1}{2}(q-s)$,

$c = \sqrt{\alpha^2+\beta^2}$, $b = \sqrt{c^2-a^2}$.

- Foci $\pm(c,0)$: $\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1$.
- Foci $\pm(\alpha,\beta)$: $X = \dfrac{\alpha U + \beta V}{\sqrt{\alpha^2+\beta^2}}$, $Y = \dfrac{-\beta U + \alpha V}{\sqrt{\alpha^2+\beta^2}}$

$$\Longrightarrow \frac{\left(\dfrac{\alpha U+\beta V}{\sqrt{\alpha^2+\beta^2}}\right)^2}{a^2} - \frac{\left(\dfrac{-\beta U+\alpha V}{\sqrt{\alpha^2+\beta^2}}\right)^2}{b^2} = 1$$

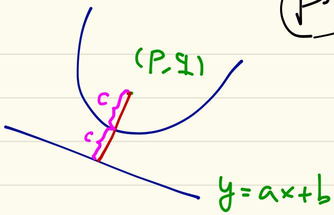
or $$(\alpha^2 b^2 - \beta^2 a^2)U^2 + 2\alpha\beta(b^2+a^2)UV + (\beta^2 b^2 - \alpha^2 a^2)V^2 = (abc)^2$$

- General foci: $U = x-p+\alpha$, $V = y-q+\beta$

$$\Rightarrow (\alpha^2 b^2 - \beta^2 a^2)(x-p+\alpha)^2 + 2\alpha\beta(b^2+a^2)(x-p+\alpha)(y-q+\beta)$$
$$+ (\beta^2 b^2 - \alpha^2 a^2)(y-q+\beta)^2 = (abc)^2.$$

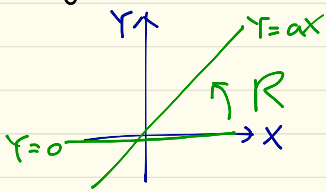
6. (i) $c = \frac{1}{2} d\left((p,q),\ y=ax+b\right)$

$$= \frac{1}{2} \cdot \frac{|ap-q+b|}{\sqrt{a^2+1}}$$



Then $Y=4cX^2+c$ be the parabola with focus $(0,2c)$ and directrix $Y=0$.

(ii) We want to rotate $Y=4cX^2+c$ into a parabola with directrix $Y=aX$ by an angle ϕ, counterclockwise.



Let $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, $\tan\phi = a$.

$\left(\cos\phi = \frac{1}{\sqrt{a^2+1}},\ \sin\phi = \frac{a}{\sqrt{a^2+1}}\right)$

$$R(X,Y) = (\underbrace{\cos\phi X - \sin\phi Y}_{U},\ \underbrace{\sin\phi X + \cos\phi Y}_{V})$$

$$\Rightarrow (X,Y) = R^{-1}(U,V) = (\cos\phi U + \sin\phi V,\ -\sin\phi U + \cos\phi V)$$

$$= \left(\frac{U+aV}{\sqrt{a^2+1}},\ \frac{-aU+V}{\sqrt{a^2+1}}\right).$$

$$\therefore\ Y = 4X^2 + c \Rightarrow \frac{-aU+V}{\sqrt{a^2+1}} = \left(\frac{U+aV}{\sqrt{a^2+1}}\right)^2 + c$$

with focus $R(0,2c) = \left(\frac{-2ac}{\sqrt{a^2+1}},\ \frac{2c}{\sqrt{a^2+1}}\right)$ and directrix $V=aU$.

(iii) For the general parabola with focus (p, q) and directrix $y = ax + b$,

Consider $T(U, V) = \left(U + \frac{2ac}{\sqrt{a^2+1}} + p,\ V - \frac{2c}{\sqrt{a^2+1}} + q\right)$

$$\Rightarrow x = U + \frac{2ac}{\sqrt{a^2+1}} + p,$$

$$y = V - \frac{2c}{\sqrt{a^2+1}} + q$$

$\Rightarrow$ the equation is

$$\frac{-a\left(x - \frac{2ac}{\sqrt{a^2+1}} + p\right) + \left(y + \frac{2c}{\sqrt{a^2+1}} + q\right)}{\sqrt{a^2+1}}$$

$$= \left[\frac{\left(x - \frac{2ac}{\sqrt{a^2+1}} + p\right) + a\left(y + \frac{2c}{\sqrt{a^2+1}} + q\right)}{\sqrt{a^2+1}}\right]^2 + c.$$