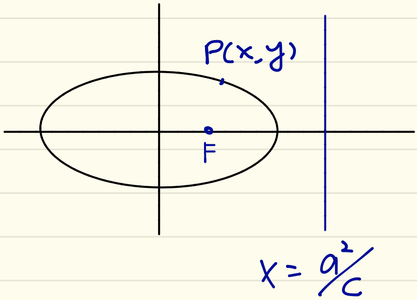
Homework 4.

1. $c = \sqrt{a^2 - b^2}$



On the right hand side,

focus $F = (c, 0)$,

directrix D: $x = a^2/c$.

Then $\overline{PD} = a^2/c - x$, $\overline{PF} = \sqrt{(x-c)^2 + y^2}$

Then $\overline{PF}^2 = (x-c)^2 + y^2 = x^2 - 2cx + c^2 + y^2$

$$\left(y^2 = b^2 - \frac{b^2}{a^2}x^2\right) \Rightarrow\ = x^2 - 2cx + c^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= \left(\frac{a^2 - b^2}{a^2}\right)x^2 - 2cx + b^2 + c^2$$

$$\left(a^2 = b^2 + c^2\right) \Rightarrow\ = \frac{c^2}{a^2}x^2 - 2cx + a^2$$

On the other hand,

$$(e\overline{PD})^2 = \left[\frac{c}{a}\left(\frac{a^2}{c} - x\right)\right]^2 = \left(a - \frac{c}{a}x\right)^2$$

$$= \frac{c^2}{a^2}x^2 - \frac{2c}{a}x + a^2 = \overline{PF}^2$$

$\therefore\ \overline{PF} = e\overline{PD}$.

On the left hand side, it is similar.

2.

(a) $6x^2+9y^2=54$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{6}=1. \quad a=3,\ b=\sqrt{6}.$$

$$c=\sqrt{9-6}=\sqrt{3}$$

eccentricity $e=c/a=\sqrt{3}/3$.

foci : $\pm(c,0)=\pm(\sqrt{3},0)$ $(\because a>b>0)$

directrics : $x=\pm\frac{a^2}{c}=\pm 3\sqrt{3}$.

(b) $9x^2+36x-16y^2=126$

$$9(x+2)^2-16y^2=126+36=162$$

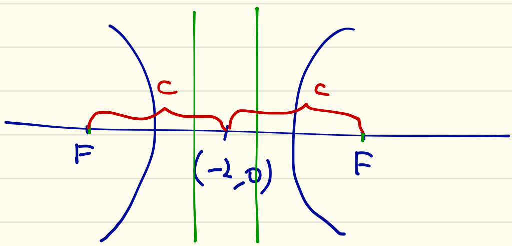
$$\Rightarrow \frac{(x+2)^2}{18}-\frac{y^2}{\cancel{81}/8}=1, \quad a=3\sqrt{2},\ b=\frac{9}{4}\sqrt{2}$$

$$c=\sqrt{a^2+b^2}=\frac{15}{4}\sqrt{2}$$

eccentricity $e=c/a=\frac{5}{4}$

foci : $(-2,0)\pm(\frac{15}{4}\sqrt{2},0)$

directrices: $x=(-2,0)\pm(\frac{a}{e},0)=(-2,0)\pm(\frac{12}{5}\sqrt{2},0)$.



(c) $y^2 - 2x = 4$

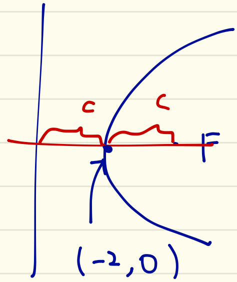
$\Rightarrow y^2 = 2x+4 = 2(x+2) = 4\left(\frac{1}{2}(x+2)\right)$

$c = \frac{1}{2}$

eccentriticy $e = 1$ ($\because$ parabola)

focus: $(-2,0) + (\frac{1}{2},0) = (\frac{-3}{2},0)$

directrix: $x = \frac{-5}{2}$

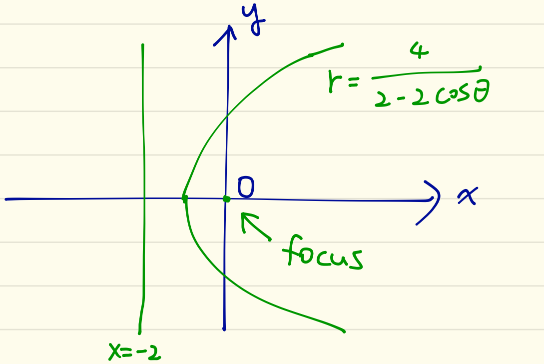


3.

(a) $r = \dfrac{4}{2-2\cos\theta} = \dfrac{2}{1-\cos\theta}$

$\Rightarrow e = 1$, a parabola. $ek = 2 \Rightarrow k = 2$

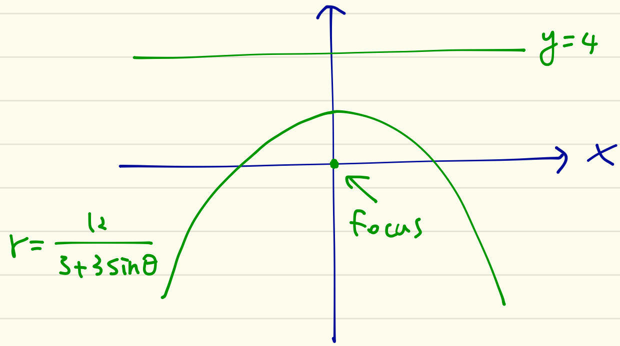
That is: focus at $(0,0)$ and directrix: $x = -2$



3. (b) $r = \dfrac{12}{3+3\sin\theta} = \dfrac{4}{1+\sin\theta}$

$\Rightarrow e = 1$, a parabola, $ek = 4 \Rightarrow k = 4$

This is: focus at $(0,0)$, directrix: $y = 4$.



(c) $r = \dfrac{25}{10-5\cos(\theta-\frac{\pi}{3})} = \dfrac{5/2}{1-\frac{1}{2}\cos(\theta-\frac{\pi}{3})}$

$\Rightarrow e = \frac{1}{2} < 1$: an ellipse, $ek = \frac{5}{2} \Rightarrow k = 5$

This is obtained from the ellipse $r = \dfrac{5/2}{1-\frac{1}{2}\cos\theta}$ by a rotation map R, which rotates every point on $\mathbb{R}^2$ by the angle $\pi/3$, counterclockwise.

For the ellipse $r = \dfrac{5/2}{1-\frac{1}{2}\cos\theta}$,

$F_1' = (0,0)$ and $x = -5$ is a pair of focus and directrix.

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We have to find the other focus and directrix.

$\because \overline{PF_1} = e\overline{PD}$, we consider the left end point $P_1(t,0)$

Then $(-t) = \frac{1}{2}(t-(-5))$

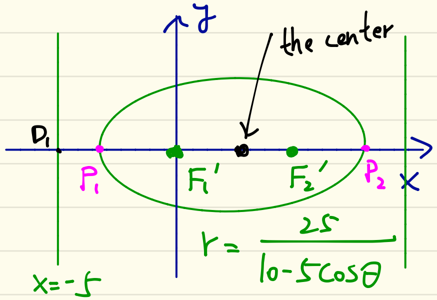
$\Rightarrow \frac{3}{2}t = \frac{-5}{2} \Rightarrow t = \frac{-5}{3}$.

Similarly, we have the right end point $P_2(5,0)$

$\therefore 2a = 5 + \frac{5}{3} = \frac{20}{3}$, $a = \frac{10}{3}$

$\Rightarrow$ the center is $(\frac{5}{3}, 0)$, $F_2' = (\frac{10}{3}, 0)$

and the other directrix is $x = \frac{10}{3} + 5 = \frac{25}{3}$.

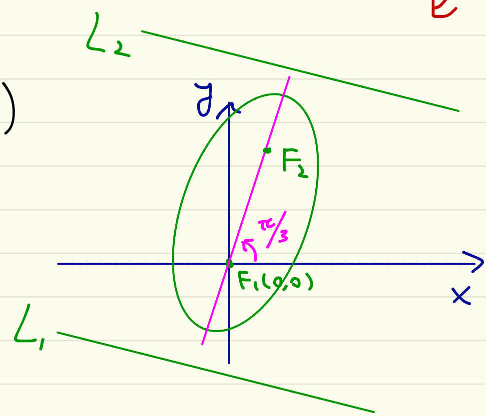


Now consider the map

$$R(x,y) = \left(x\cos\frac{\pi}{3} - y\sin\frac{\pi}{3},\ x\sin\frac{\pi}{3} + y\cos\frac{\pi}{3}\right)$$

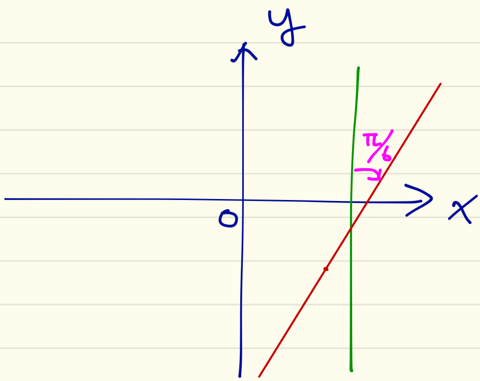
$$= \left(\frac{x - \sqrt{3}y}{2}, \frac{\sqrt{3}x + y}{2}\right)$$

$$\Rightarrow \begin{cases} F_1 = (0,0), \\ F_2 = R(F_2') = R(\frac{10}{3}, 0) \\ \quad = (\frac{5}{3}, \frac{5\sqrt{3}}{3}) \\ L_1 : \frac{1}{2}(x + \sqrt{3}y) = -5, \\ L_2 : \frac{1}{2}(x + \sqrt{3}y) = \frac{25}{3}. \end{cases}$$



3. (d) $r\cos\left(\theta+\frac{\pi}{6}\right)=2$ is obtained from the curve $r\cos\theta=2$ from a rotation map R, which rotates every point on $\mathbb{R}^2$ by the angle $\pi/6$, clockwise.

Note that $r\cos\theta=2$ is the straight line $x=2$



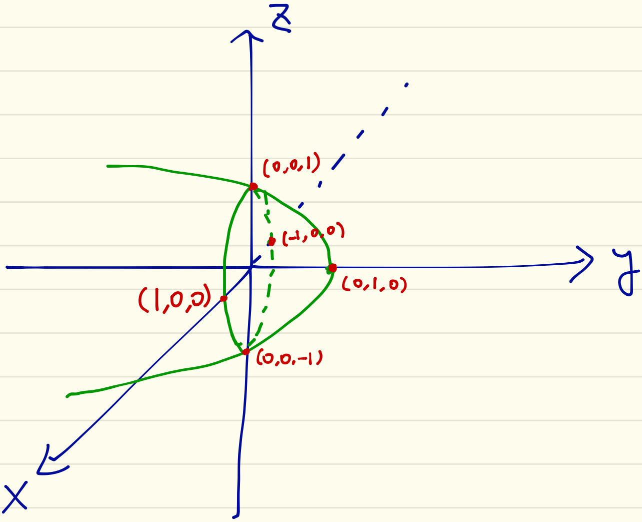
$r\cos\left(\theta+\frac{\pi}{6}\right)=2$. $\quad x=2$

$\Updownarrow$

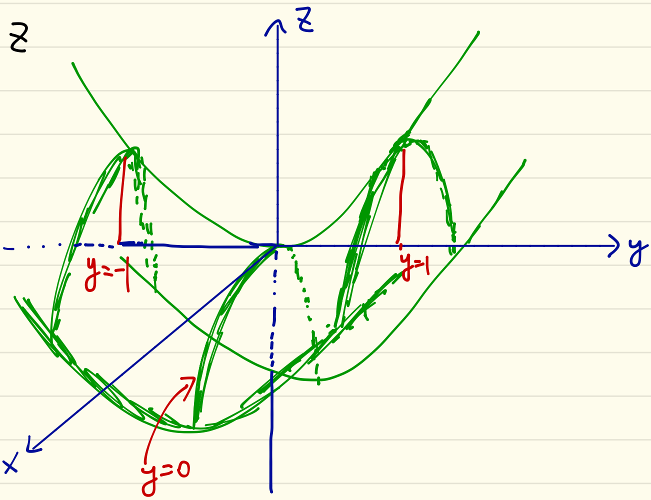
$\frac{\sqrt{3}}{2}x-\frac{1}{2}y=2$.

Problems concerning quadric surfaces:

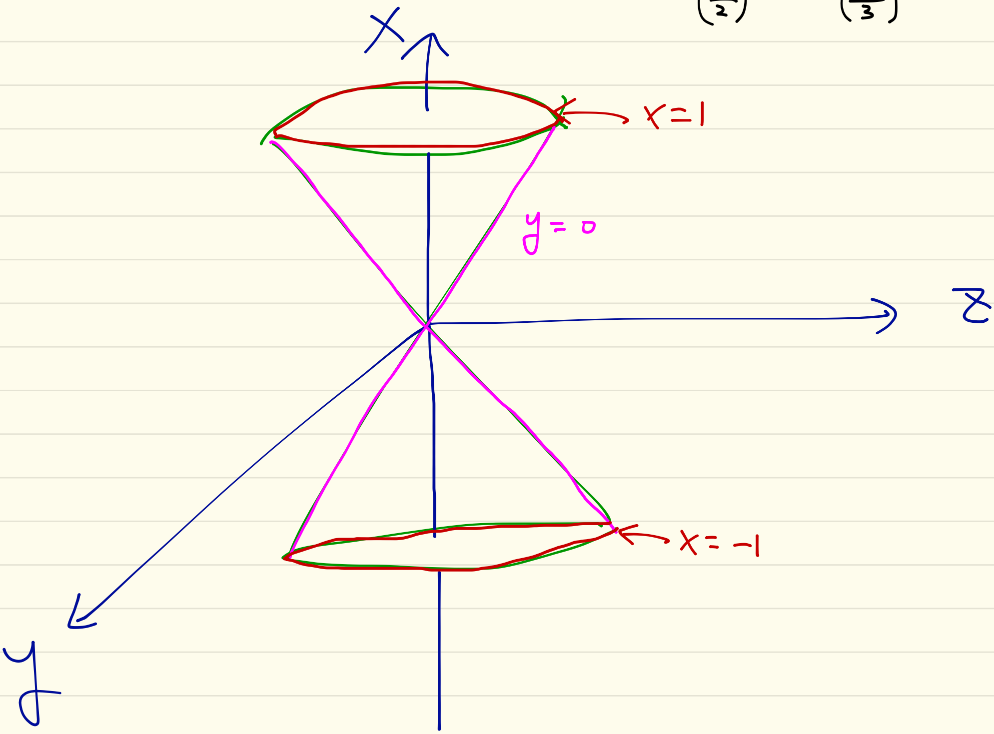
1. (a) $x^2 = 1 - y - z^2 \iff x^2 + z^2 = 1 - y$



(b) $y^2 - x^2 = z$



1. (c) $16y^2 + 9z^2 = 4x^2$. $\Rightarrow$ $x \neq 0$: $\dfrac{y^2}{\left(\frac{x}{2}\right)^2} + \dfrac{z^2}{\left(\frac{2x}{3}\right)^2} = 1$



2. (a) is a graph of the function $y = f(x,z) = 1 - x^2 - z^2$.

(b) is a graph of the function $z = f(x,y) = y^2 - x^2$.

(c) is not a graph of a function.