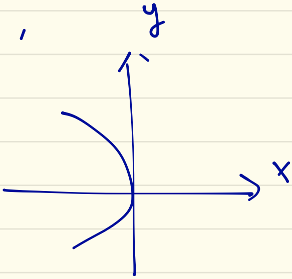
Homework 5

Determination of traces:

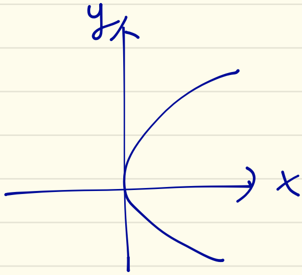
1. (a) $x = 2z^2 - 2y^2$: its trace is either (k) or (l)

fix $z=0$,

(k):



(l)



So its trace is (k).

(b) $9x^2+4y^2+2z^2=36$ its trace is either (c) or (d).

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{(3\sqrt{2})^2} = 1$$

The endpoints are $(\pm 2,0,0)$, $(0,\pm 3,0)$, $(0,0,\pm 3\sqrt{2})$

$3\sqrt{2} > 3 > 2 \Rightarrow$ (c)

(c) $x = -y^2 - z^2$: $x \le 0 \ \forall y, z$.

$\Rightarrow$ (e)

(d) $z = -2x^2 - y^2$: $z \le 0 \ \forall x, y$

$\Rightarrow$ (f).

More change coordinates:

1. (a) $9x^2+16y^2+4z^2-8z-140\le 0$, $z\ge 1$

$\Rightarrow 9x^2+16y^2+4(z-1)^2\le 144$, $z\ge 1$

$\Rightarrow \dfrac{x^2}{16}+\dfrac{y^2}{9}+\dfrac{(z-1)^2}{36}\le 1$, $z\ge 1$

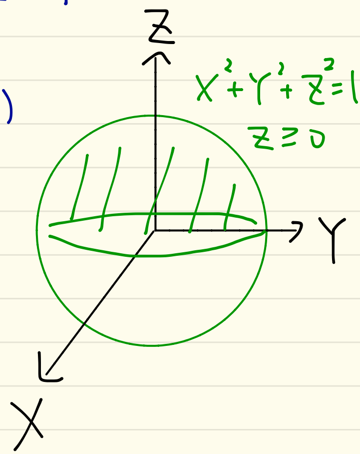
Let $X=\frac{1}{4}x$, $Y=\frac{1}{3}y$, $Z=\frac{1}{6}(z-1)$

Then the region bounded by

$\frac{x^2}{16}+\frac{y^2}{9}+\frac{(z-1)^2}{36}\le 1$ and $z\ge 1$

turns into the region bounded by

$X^2+Y^2+Z^2\le 1$ and $Z\ge 0$.



By spherical coordinate, the region is

$X=\rho\sin\phi\cos\theta$, $Y=\rho\sin\phi\sin\theta$, $Z=\rho\cos\phi$,

for $0\le\rho\le 1$, $0\le\phi\le\pi/2$, $0\le\theta<\pi$.

$\Rightarrow x=4\rho\sin\phi\sin\theta$,

$y=3\rho\sin\phi\sin\theta$,

$z=6\rho\cos\phi+1$.

1. (b) $y = 3x + 8$, $y = -2x + 7$, $y = 3x + 12$, $y = -2x + 10$

$\Rightarrow y - 3x = 8$, $y + 2x = 7$, $y - 3x = 12$, $y + 2x = 10$.

$\therefore$ The region is $\begin{cases} 8 \le y - 3x \le 12 \\ 7 \le y + 2x \le 10. \end{cases}$

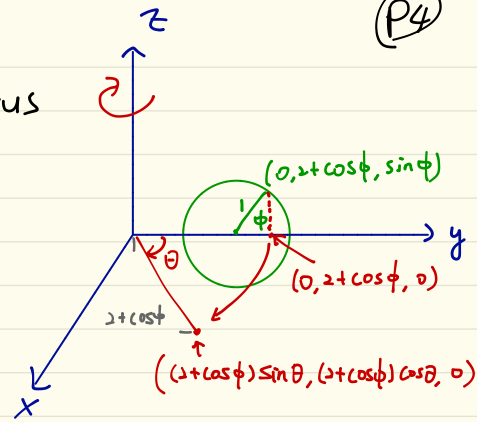
Let $u = y - 3x$, $v = y + 2x$, then we turn the original region into a retangle

$$\begin{cases} 8 \le u \le 12 \\ 7 \le v \le 10 \end{cases} .$$

2. (a) Any point in the torus can be obtained by the following two steps:

① Take a point in the "green circle"

② Rotate the point by θ, around z-axis, $\theta \in [0, 2\pi)$



Any point in the "green circle" is of the form $(0,\ 2+\cos\phi,\ \sin\phi)$, $\phi \in [0, 2\pi)$

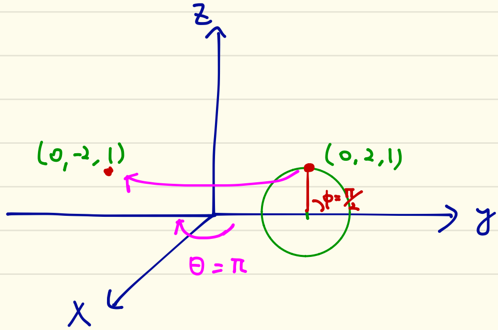
$\Rightarrow$ Rotate the point by θ, around z-axis, we get the point $((2+\cos\phi)\sin\theta,\ (2+\cos\phi)\cos\theta,\ \sin\phi)$

$\theta, \phi \in [0, 2\pi)$

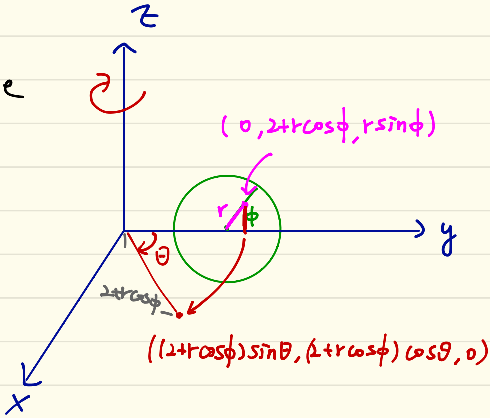
(b) $\theta = \pi$, $\phi = \pi/2$

$\Rightarrow \left(\left(2+\cos\frac{\pi}{2}\right)\sin\pi,\ \left(2+\cos\frac{\pi}{2}\right)\cos\pi,\ \sin\frac{\pi}{2} \right)$

$= (0, -2, 1)$



(c) Take a point bounded by the "green circle" on yz-plane $(0, 2+r\cos\phi, r\sin\phi)$, $\phi \in [0, 2\pi)$, $r \in [0,1]$



Rotate the point by θ, around z-axis, we obtain the point

$$((2+r\cos\phi)\sin\theta, (2+r\cos\phi)\cos\theta, r\sin\phi),$$

$$\theta, \phi \in [0, 2\pi), \ r \in [0,1].$$

Vectors in $\mathbb{R}^3$:

1. Let p, q be two distinct points. Note that $L: p + \mathbb{R}(q-p)$ is a line passing through p and q.
Let $L': p' + \mathbb{R}q'$ be another line passing p and q.
It suffices to show that $L' = L$.

$\left(\Leftrightarrow \underline{\mathbb{R}q' = \mathbb{R}(q-p)}^{②} \text{ and } \underline{p'-p \in \mathbb{R}q'}^{①}\right)$

↳ Prop 1.1. in Notes of Smith.

①: $\because p \in L'$, $\therefore \exists\, t \in \mathbb{R}$ s.t. $p = p' + tq' \Rightarrow p' - p = -tq' \in \mathbb{R}q'$.

②: $\because q \in L'$, $\therefore \exists\, s \in \mathbb{R}$ s.t. $q = p' + sq'$

$$\Rightarrow q - p = p' + sq' - p = (s-t)q'$$

Then $\forall\, x = a(q-p) \in \mathbb{R}(q-p)$,

$$x = a(s-t)q' \in \mathbb{R}q' \quad \therefore \mathbb{R}(q-p) \subseteq \mathbb{R}q'.$$

On the other hand, $q \neq p \Rightarrow q - p \neq 0 \Rightarrow s - t \neq 0$

$$\therefore q' = \frac{1}{s-t}(q-p).$$

$$\therefore \forall\, y = bq' \in \mathbb{R}q', \quad y = \frac{b}{s-t}(q-p) \in \mathbb{R}(q-p)$$

$$\therefore \mathbb{R}(q-p) \supseteq \mathbb{R}q'.$$

$$\therefore \mathbb{R}q' = \mathbb{R}(q-p).$$

$\therefore$ By prop. $L' = L$.

Let L be the line $(1,-1,0)+\mathbb{R}(1,1,1)$, and

L' be the line given by $\begin{cases} x+y-2z=0 \\ 2x-y-z=3 \end{cases}$.

- $(1,-1,0)$ and $(2,0,1)$ are two different points in L.

- $\begin{cases} 1+(-1)-2\cdot 0=0 \\ 2\cdot 1-(-1)-0=3 \end{cases}$, $\begin{cases} 2+0-2\cdot 1=0 \\ 2\cdot 2-0-1=3 \end{cases}$

$\Rightarrow$ $(1,-1,0)$ and $(2,0,-1)$ are also in L'.

$\therefore$ $(1,-1,0)$ and $(2,0,-1)$ $\in L\cap L'$ $\Rightarrow$ $L=L'$.