Homework 6

Subspaces and Spans:

1. (a) $W = \{(x,y,z) \mid x=3y, z=-y\}$

- $\because 0 = 3\cdot 0,\ 0 = -0 \quad \therefore (0,0,0) \in W \Rightarrow W \neq \phi$
- If $(x,y,z), (x',y',z') \in W$, then $x=3y,\ z=-y$ and $x'=3y',\ z'=-y' \Rightarrow x+x' = 3(y+y'),\ z+z' = -(y+y')$

  $\therefore (x,y,z)+(x',y',z') = (x+x', y+y', z+z') \in W$
- If $(x,y,z) \in W$, $c \in \mathbb{R}$, then $x=3y,\ z=-y$

  $\Rightarrow cx = 3(cy),\ cz = -(cy)$

  $\therefore c(x,y,z) = (cx, cy, cz) \in W \quad \therefore W$ is a subspace.

(b) $W = \{(x,y,z) \mid x = z+2\}$

For $(x,y,z) = (2,0,0) \in W$, $c = 2 \in \mathbb{R}$

$c(x,y,z) = (4,0,0) \notin W \quad \therefore W$ is not a subspace

(c) $W = \{(x,y,z) \mid 2x - y + z = 0\}$

- $\because 2\cdot 0 - 0 + 0 = 0, \quad \therefore (0,0,0) \in W \Rightarrow W \neq \phi.$
- If $(x,y,z), (x',y',z') \in W$, then

  $2x - y + z = 0$ and $2x' - y' + z' = 0$

  $\Rightarrow 2(x+x') - (y+y') + (z+z') = 0$

  $\therefore (x,y,z) + (x',y',z') = (x+x', y+y', z+z') \in W$
- If $(x,y,z) \in W$, $c \in \mathbb{R}$, then $2x - y + z = 0$

  $\Rightarrow 2(cx) - (cy) + (cz) = c\cdot 0 = 0$

  $\therefore c(x,y,z) = (cx, cy, cz) \in W. \quad \therefore W$ is a subspace.

1. (d) $W = \{(x,y,z) \mid x+6y-9z=1\}$

Consider $(x,y,z) = (1,0,0) \in W$, $c=2 \in \mathbb{R}$

$c\cdot(x,y,z) = (2,0,0) \notin W$ $\therefore$ not subspace.

(e) $W = \{(x,y,z) \mid x^2+y=0\}$

$(x,y,z) = (1,-1,0) \in W$, $c=2 \in \mathbb{R}$

$c\cdot(x,y,z) = (2,-2,0) \notin W$. $\therefore$ not subspace.

2. (a) • $U \cap V \neq \phi$ since $0 \in U$ and $0 \in V$.

• If $u, v \in U \cap V$, then

$u, v \in U \Rightarrow u+v \in U$, $\because U$: subspace

$u, v \in V \Rightarrow u+v \in V$ $\because V$: subspace

$\therefore u+v \in U \cap V$.

• If $u \in U \cap V$, $c \in \mathbb{R}$, then

$u \in U \Rightarrow cu \in U$, $\because U$: subspace

$u \in V \Rightarrow cu \in V$ $\because V$: subspace

$\therefore cu \in U \cap V$ $\therefore U \cap V$ is a subspace.

2 (b) • $\because 0 \in U,\ 0 \in V \Rightarrow 0 = 0 + 0 \in U+V$

$\therefore U+V \neq \phi$

• If $u, v \in U+V$, then $u = u_1 + u_2$, $v = v_1 + v_2$ for some $u_1, v_1 \in U$, $u_2, v_2 \in V$

$$\Rightarrow u + v = (u_1 + v_1) + (u_2 + v_2)$$

$\because U, V$: subspace

$$= (u_1 + u_2) + (v_1 + v_2) \in U+V.$$

• If $u \in U+V$, $c \in \mathbb{R}$, then $u = u_1 + u_2$ for some $u_1 \in U$, $u_2 \in V$.

$$\Rightarrow cu = c(u_1 + u_2) = cu_1 + cu_2 \in U+V$$

$\therefore U+V$ is a subspace of $\mathbb{R}^n$.

(c) $U \cup V$ is not necessary a subspace.

For example, in $\mathbb{R}^3$,

Consider $U = \{(x, y, z) \mid y = z = 0\}$,

$$V = \{(x, y, z) \mid x = z = 0\}$$

$\Rightarrow (1, 0, 0) \in U$, $(0, 1, 0) \in V$,

but $(1,0,0) + (0,1,0) = (1,1,0) \notin U, V$

$\Rightarrow (1,0,0) + (0,1,0) \notin U \cup V$

$\therefore U \cup V$ is not a subspace in the example.

3. (a) If $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = a\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} a+2b \\ 3a+4b \\ b \end{pmatrix}$

$\Rightarrow b=3$. $3a+4b=0 \Rightarrow a=-4$.

but in this solution, $a+2b=-4+2\cdot 3=2 \neq -2$

$\therefore$ the first vector is not in the span of the other two vectors.

(b) If $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a-2b \\ -2a-b \\ a+b \end{pmatrix}$

$$\Rightarrow \begin{cases} a-2b=3 & — ① \\ -2a-b=4 & — ② \\ a+b=1 & — ③ \end{cases}$$

②+③ : $-a=5$, $a=-5$ — ④

by ③ & ④ : $b=6$

but $a-2b=-5-2\cdot 6=-17 \neq 3$

$\therefore$ the first vector is not in the span of the other two vectors.

(c) If $\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} a-3b \\ 2a-3b \\ -a+3b \end{pmatrix}$

$$\Rightarrow \begin{cases} a-3b=-2 \\ 2a-3b=2 \\ -a+3b=2 \end{cases} \Rightarrow \quad a=4,\ b=2$$

$$\Rightarrow \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}.$$

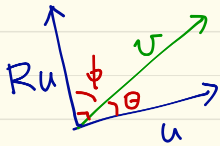
Plane Geometry

1. $u = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$ $\Rightarrow$ $u \times v = \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$

$\therefore |u \times v| = |u_1 v_2 - u_2 v_1|$.

On the other hand, we want to compute $|u| \cdot |v| \cdot \sin\theta$.

Let R be the map rotats every point on xy-plane by the angle $\frac{\pi}{2}$, counterclockwise.



$$\Rightarrow R\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x\cos\frac{\pi}{2} - y\sin\frac{\pi}{2} \\ x\sin\frac{\pi}{2} + y\cos\frac{\pi}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} .$$

Let ϕ be the angle between $R(Ru)$ and Rv .

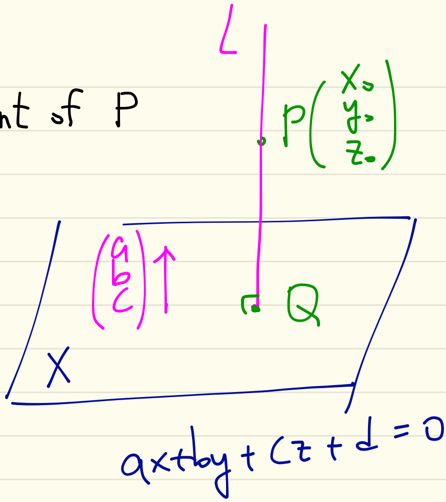
Then we have that $\sin\theta = |\cos\phi|$.

$$\therefore |u| \cdot |v| \sin\theta = |Ru| \cdot |v| \cdot |\cos\phi| = |Ru \cdot v|$$

$$= \left| \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \right| = |u_1 v_2 - u_2 v_1| .$$

So $|u \times v| = |u| \cdot |v| \sin\theta$ is true for vectors on xy-plane.

2. Let Q be the projection point of P on the plane X.



The equation of L is as following:

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$

Then Q is the intersection point of L and X. So we have to find the solution of the simultaneous equations :
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \\ ax+by+cz+d=0 \end{cases}$$

$$\Rightarrow t = t_1 = \frac{-ax_0 - by_0 - cz_0 - d}{a^2+b^2+c^2}, \quad Q = \begin{pmatrix} x_0 + at_1 \\ y_0 + bt_1 \\ z_0 + ct_1 \end{pmatrix}.$$

$\Rightarrow$ The distince is $|\overline{PQ}|$

$$= \sqrt{(x_0+at_1-x_0)^2 + (y_0+bt_1-y_0)^2 + (z_0+ct_1-z_0)^2}$$

$$= \sqrt{(at_1)^2 + (bt_1)^2 + (ct_1)^2} = |t_1| \cdot \sqrt{a^2+b^2+c^2}$$

$$= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}.$$