Name and Student ID: ______________________________

Homework 7, Analytic Geometry and Matrices

Linear (In-)dependence

1. For the following sets of vectors, determine all values of a so that they are linearly dependent:
   (a) $\{(1,1)^T, (2,a)^T\}$
   (b) $\{(1.a)^T, (a,4)^T\}$
   (c) $\{(1,1,1)^T, (1,2,-1)^T, (1,-1,a)^T\}$

2. Prove that a set of mutually orthogonal nonzero vectors, that is, $\{v_1, \ldots, v_m \mid v_i \neq 0 \text{ and } v_i \cdot v_j = 0 \text{ if } i \neq j\}$, are linearly independent.

Basis, Dimension, and Coordinate Representation

1. Let $U \subset \mathbb{R}^5$ be the solution space to the system of equations

$$\begin{cases} 3x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ 2x_1 + 5x_2 - x_3 + x_4 - x_5 = 0 \\ 3x_1 + x_2 - 2x_3 - x_4 - x_5 = 0. \end{cases}$$

   (a) U is a subspace of $\mathbb{R}^5$.
   (b) Find the dimension of U.
   (c) Find a basis of U.

2. Prove the converse of Theorem 2 in class. That is, if $B = \{v_1, \ldots, v_r\} \subset W$ is a subset of the subspace W so that every element of W can be written uniquely as a linear combination of elements in B, then B is a basis of W. (This equivalent condition is then used as definition for basis when the vector space is of infinite dimension.)

3. Consider the subset $B = \{(1,2,1)^T, (2,1,1)^T, (3,0,3)^T\} \subset \mathbb{R}^3$.
   (a) Prove that B forms a basis of $\mathbb{R}^3$.
   (b) Write down the coordinate representation of $e_1, e_2, e_3,$, the standard basis vectors for $\mathbb{R}^3$, with respect to the basis B.
   (c) Write down the coordinate representation of a general vector v, with standard coordinate representation $(x,y,z)^T$, with respect to the basis B.