Homework 7.

Linear (In-) dependence

1. (a) $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\a\end{pmatrix}\right\}$: linearly dependent

$\Leftrightarrow \begin{pmatrix}2\\a\end{pmatrix} \in sp\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \Leftrightarrow \begin{pmatrix}2\\a\end{pmatrix} = t\begin{pmatrix}1\\1\end{pmatrix}$ for some $t \in \mathbb{R}$

$\Leftrightarrow a = 2.$

(b) $\left\{\begin{pmatrix}1\\a\end{pmatrix}, \begin{pmatrix}a\\4\end{pmatrix}\right\}$: linearly dependent

$\Leftrightarrow \begin{pmatrix}a\\4\end{pmatrix} \in sp\left\{\begin{pmatrix}1\\a\end{pmatrix}\right\} \Leftrightarrow \begin{pmatrix}a\\4\end{pmatrix} = t\begin{pmatrix}1\\a\end{pmatrix}$ for some $t \in \mathbb{R}$

$\Leftrightarrow \begin{cases} a = t \\ 4 = at \end{cases}$ has solutions $\Leftrightarrow a^2 = 4 \Leftrightarrow a = \pm 2.$

(c) We use the theorem :

$\{v_1, v_2, v_3\}$ is linearly dependent

$\Leftrightarrow \exists\, (m,n,t) \neq (0,0,0)$ such that $mv_1 + nv_2 + tv_3 = 0.$

$$m\begin{pmatrix}1\\1\\1\end{pmatrix} + n\begin{pmatrix}1\\2\\-1\end{pmatrix} + t\begin{pmatrix}1\\-1\\a\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \begin{cases} m+n+t = 0 \\ m+2n-t = 0 \\ m-n+at = 0 \end{cases}$$

Then $m = -3t,\ n = 2t,\ -5t + at = 0 \Rightarrow at = 5t$

$\therefore\ a = 5 \Leftrightarrow (m,n,t)$ has non-zero solutions

$\Leftrightarrow \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\a\end{pmatrix}\right\}$ is linearly dependent.

2. If $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$,

then $\forall i$,

$$(a_1v_1 + a_2v_2 + \cdots + a_mv_m)\cdot v_i = a_i(v_i \cdot v_i) = 0$$

$$v_i \neq 0 \Rightarrow v_i \cdot v_i \neq 0 \Rightarrow a_i = 0$$

$\therefore$ $v_1, \cdots, v_m$ are linearly independent.

Basis, Dimension, and Coordinate Representation.

1. $$\begin{cases} 3x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ 2x_1 + 5x_2 - x_3 + x_4 - x_5 = 0 \\ 3x_1 + x_2 - 2x_3 - x_4 - x_5 = 0 \end{cases} \quad (*)$$

(a)

Clearly $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $(*)$.

For any $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} \in U$, $a \in \mathbb{R}$.

check $u+v$ and au satisfy $(*)$ yourself.

$\therefore$ U is a subspace of $\mathbb{R}^5$.

1. (b) Solve $(*)$, we get
   (c) $X_3 = 6X_1 + 2X_2$, $X_4 = \frac{-5}{2}X_1 + 3X_2$, $X_5 = \frac{-13}{2}X_1$

$$\Rightarrow U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 6x_1 + 2x_2 \\ \frac{-5}{2}x_1 + 3x_2 \\ \frac{-13}{2}x_1 \end{pmatrix} \middle| \, x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 2 \\ 0 \\ 12 \\ -5 \\ -13 \end{pmatrix} \cdot \frac{x_1}{2} + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} x_2 \, \middle| \, x_1, x_2 \in \mathbb{R} \right\}$$

$$= Sp \left\{ \underbrace{\begin{pmatrix} 2 \\ 0 \\ 12 \\ -5 \\ -13 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}}_{v_2} \right\}$$

It is easy to check that $a_1 v_1 + a_2 v_2 = 0 \rightarrow a_1 = a_2 = 0$

So $\{v_1, v_2\}$ forms a basis for U and $\dim U = 2$.

2. $B = \{v_1, \cdots, v_r\} \subseteq W$.

Since every element of W can be written (uniquely) as a linear combination of element in B, we have $\operatorname{span} B = W$.

Now we show that B is linearly independent:

If $a_1 v_1 + \cdots + a_r v_r = 0$,

$\because$ $0 = 0v_1 + \cdots + 0v_r$ is the unique representation,

$\therefore$ $a_1 = a_2 = \cdots = a_r = 0$.

So B is a basis for W.

3. (a) Span:

$\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

for $a = \frac{1}{6}(-3x+6y-z)$, $b = \frac{1}{6}(3x-z)$, $c = \frac{1}{2}(x-2y+z)$.

$\therefore$ $\operatorname{sp} B = \mathbb{R}^3$.

Linearly independent:

If $a_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + a_2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} a_1 + 2a_2 + 3a_3 = 0 \\ 2a_1 + a_2 = 0 \\ a_1 + a_2 + 3a_3 = 0 \end{cases}$ $\Rightarrow a_1 = a_2 = a_3 = 0$

$\therefore$ B is a basis for $\mathbb{R}^3$.

3\. (b) $$e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}=a_1\begin{pmatrix}1\\2\\1\end{pmatrix}+a_2\begin{pmatrix}2\\1\\1\end{pmatrix}+a_3\begin{pmatrix}3\\0\\3\end{pmatrix}$$

$$\Rightarrow a_1=-\frac{1}{2},\quad a_2=1,\quad a_3=-\frac{1}{6}$$

$$\therefore [e_1]_B=\begin{pmatrix}-\frac{1}{2}\\1\\-\frac{1}{6}\end{pmatrix}.$$

$$e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}=b_1\begin{pmatrix}1\\2\\1\end{pmatrix}+b_2\begin{pmatrix}2\\1\\1\end{pmatrix}+b_3\begin{pmatrix}3\\0\\3\end{pmatrix}$$

$$\Rightarrow b_1=\frac{1}{2},\quad b_2=0,\quad b_3=-\frac{1}{6}$$

$$\therefore [e_2]_B=\begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{6}\end{pmatrix}$$

$$e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}=c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}2\\1\\1\end{pmatrix}+c_3\begin{pmatrix}3\\0\\3\end{pmatrix}$$

$$\Rightarrow c_1=\frac{1}{2},\quad c_2=-1,\quad c_3=\frac{1}{2}$$

$$\therefore [e_3]_B=\begin{pmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{pmatrix}.$$

(c) $$\left[\begin{pmatrix}x\\y\\z\end{pmatrix}\right]_B=[xe_1+ye_2+ze_3]_B=x[e_1]_B+y[e_2]_B+z[e_3]_B$$

$$=x\begin{pmatrix}\frac{-1}{2}\\1\\\frac{-1}{6}\end{pmatrix}+y\begin{pmatrix}\frac{1}{2}\\0\\\frac{-1}{6}\end{pmatrix}+z\begin{pmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(-x+y+z)\\x-z\\\frac{-1}{6}(x+y-3z)\end{pmatrix}.$$