Name and Student ID: ____________________

Homework 8, Analytic Geometry and Matrices

1. Determine, with sufficient reasons, whether the following linear transformations are linear.

   (a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ z-y \\ x+y+2 \end{pmatrix}$.

   (b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$ for some $\phi \in \mathbb{R}$.

2. Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

   Find $T\begin{pmatrix} 8 \\ 11 \end{pmatrix}$.

3. Find the nullity and rank of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ 2z \end{pmatrix}.$$

4. Given a linear transformation $T : V \to W$. Prove that

   (a) If $\dim V > \dim W$, T can not be one-to-one.

   (b) If $\dim V < \dim W$, T can not be onto.

5. Determine if the following maps are onto.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ x+2y \\ 2y-4x \end{pmatrix}.$$

   (b) $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+z \\ y \\ 2z+w \\ x-w \end{pmatrix}.$$

A *direct sum* decomposition of a vector space V consists of two subspaces W_1, W_2 of V, so that $W_1 \cap W_2 = \{0\}$ and $V = W_1 + W_2$. Recall that

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}.$$

This decomposition of V is denoted by

$$V = W_1 \oplus W_2.$$

6. Prove that if $V = W_1 \oplus W_2$, every element $v \in V$ can be written *uniquely* as $v = w_1 + w_2$, where $w_i \in W_i$.

7. On $\mathbb{R}^4$, let $W_1 = Sp((1,0,0,0)^T, (0,1,0,0)^T)$ and $W_2 = Sp((1,1,1,1)^T, (0,1,1,-1)^T)$.
   (a) Prove that $\mathbb{R}^4 = W_1 \oplus W_2$.
   (b) Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(v) = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$ is the direct sum decomposition of v defined above. Explain why is T well defined, that is, why does it return only one output for each input.
   (c) Write down $[T]_\beta^\gamma$, where $\beta = \{e_1, e_2, e_3, e_4\}$ is the standard bases of $\mathbb{R}^4$ and $\gamma = \{e_1, e_2, (1,1,1,1)^T, (0,1,1,-1)^T\}$.
   (d)
$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} =?$$
   (e) Prove that T is an isomorphism.