Homework 8

1. (a) No.

Since $T\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$

$$\Rightarrow T\left(\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}\right) \neq T\begin{pmatrix}0\\0\\0\end{pmatrix}+T\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

(b) For all $\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x'\\y'\end{pmatrix} \in \mathbb{R}^2$, $a \in \mathbb{R}$

$$T\left(\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}x'\\y'\end{pmatrix}\right) = T\begin{pmatrix}x+x'\\y+y'\end{pmatrix} = \begin{pmatrix}(x+x')\cos\phi-(y+y')\sin\phi\\(x+x')\sin\phi+(y+y')\cos\phi\end{pmatrix}$$

$$= \begin{pmatrix}x\cos\phi-y\sin\phi\\x\sin\phi+y\cos\phi\end{pmatrix}+\begin{pmatrix}x'\cos\phi-y'\sin\phi\\x'\sin\phi+y'\cos\phi\end{pmatrix}$$

$$= T\begin{pmatrix}x\\y\end{pmatrix}+T\begin{pmatrix}x'\\y'\end{pmatrix}$$

$$T\left(a\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}ax\\ay\end{pmatrix} = \begin{pmatrix}ax\cos\phi-ay\sin\phi\\ax\sin\phi+ay\cos\phi\end{pmatrix}$$

$$= a\begin{pmatrix}x\cos\phi-y\sin\phi\\x\sin\phi+y\cos\phi\end{pmatrix} = aT\begin{pmatrix}x\\y\end{pmatrix}.$$

$\therefore T$ is a linear transformation.

2. $\begin{pmatrix}8\\11\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}2\\3\end{pmatrix}$, solve a, b:

$$\begin{cases} a+2b=8 \\ a+3b=11 \end{cases} \Rightarrow a=2,\ b=3$$

$$\therefore T\begin{pmatrix}8\\11\end{pmatrix} = T\left(2\begin{pmatrix}1\\1\end{pmatrix}+3\begin{pmatrix}2\\3\end{pmatrix}\right) = T\left(2\begin{pmatrix}1\\1\end{pmatrix}\right)+T\left(3\begin{pmatrix}2\\3\end{pmatrix}\right)$$

$$= 2T\begin{pmatrix}1\\1\end{pmatrix} + 3T\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\\2\end{pmatrix} + 3\begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}5\\-3\\16\end{pmatrix}.$$

3. First, find the null space $N(T) = \{v \in \mathbb{R}^3 \mid T(v) = 0\}$

If $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{cases} x-y=0 \\ 2z=0 \end{cases} \Rightarrow y=x,\ z=0$

$$\therefore N(T) = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y=x, z=0\right\} = \left\{\begin{pmatrix}x\\x\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}.$$

$\therefore$ nullity $T = \dim N(T) = 1$.

By the dimension theorem:

$$\dim(\mathbb{R}^3) = \text{nullity}\,T + \text{rank}\,T,$$

we have that $\text{rank}\,T = \dim(\mathbb{R}^3) - \text{nullity}\,T$

$$= 3 - 1 = 2.$$

4. $T: V \longrightarrow W.$

Notice that :

T is one-to-one $\Leftrightarrow N(T) = \{0\} \Leftrightarrow \text{nullity}(T) = 0,$

T is onto $\Leftrightarrow R(T) = W \Leftrightarrow \text{rank}(T) = \dim W.$

(a) If $\dim V > \dim W$, then $\dim V > \dim R(T)$
$(\because R(T) \leq W)$

$\therefore \text{nullity}(T) = \dim V - \text{rank}\, T > 0$

$\therefore$ T is not one-to-one.

(b) If $\dim V < \dim W$, then

$\text{rank}(T) = \dim V - \text{nullity}(T) \leq \dim V < \dim V$

$\therefore$ T is not onto.

5. (a) $V = \mathbb{R}^2$, $W = \mathbb{R}^4$, $T: V \longrightarrow W$ and $\dim V < \dim W$, by 4. (b), T is not onto.

5. (b) Notice : $T : \underline{\underline{V}} \longrightarrow \underline{\underline{V}}$, 要相同才能這樣做.

$$\dim V = \text{nullity}(T) + \text{rank}(T).$$

「 $\text{nullity}(T) = 0 \iff \text{rank}(T) = \dim V$ 」

$\therefore$ T is one-to-one $\iff$ T is onto.

Compute the nullspace $N(T)$:

$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + z = 0 \\ y = 0 \\ 2z + w = 0 \\ x - w = 0 \end{cases} \Rightarrow x = y = z = w = 0$$

$\therefore N(T) = \{0\}$

$\therefore \text{nullity}(T) = 0$

$\therefore T$ is one-to-one

$\therefore T$ is onto.

6. $V = W_1 \oplus W_2 \iff \begin{cases} W_1 + W_2 = V & \text{——— ①} \\ W_1 \cap W_2 = \{0\} & \text{——— ②} \end{cases}$

- $\forall\, v \in V$, $\exists\, w_1 \in W_1$, $w_2 \in W_2$ such that $v = w_1 + w_2$ by ①.

- Given $v \in V$. If $\exists\, w_1, w_1' \in W_1$, $w_2, w_2' \in W_2$ such that $v = w_1 + w_2 = w_1' + w_2'$, then we have $\underbrace{w_1 - w_1'}_{\in W_1} = \underbrace{w_2' - w_2}_{\in W_2} \in W_1 \cap W_2$.

by ②, $W_1 \cap W_2 = \{0\} \Rightarrow w_1 - w_1' = w_2' - w_2 = 0$

$\Rightarrow w_1 = w_1'$ and $w_2 = w_2'$.

$\therefore$ For any element $v \in V$ can be written uniquely as $v = w_1 + w_2$ for $w_i \in W_i$.

#7. (a) We have to show:

(i) $W_1 + W_2 = \mathbb{R}^4$

(ii) $W_1 \cap W_2 = \{0\}$.

For (i), for any $(x,y,z,w)^T \in \mathbb{R}^4$,

we have to find $v_1 \in W_1$, $v_2 \in W_2$

such that $(x,y,z,w)^T = v_1 + v_2$.

$\because v_1 \in W_1$, $v_1 = a_1(1,0,0,0)^T + a_2(0,1,0,0)^T$

$v_2 \in W_2$, $v_2 = a_3(1,1,1,1)^T + a_4(0,1,1,-1)^T$.

$\therefore$ We want to solve the equation

$$(x,y,z,w) = a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(1,1,1,1) + a_4(0,1,1,-1)$$

for a_1, a_2, a_3, and a_4.

$\Rightarrow a_1 = \frac{1}{2}(2x-z-w)$, $a_2 = y-z$,

$a_3 = \frac{1}{2}(z+w)$, $a_4 = \frac{1}{2}(z-w)$.

$\therefore \mathbb{R}^4 \subseteq W_1 + W_2$. So $W_1 + W_2 = \mathbb{R}^4$.

For (ii). If $v \in W_1 \cap W_2$, then

$v = a(1,0,0,0) + b(0,1,0,0)$ $(\because v \in W_1)$

$= c(1,1,1,1) + d(0,1,1,-1)$ $(\because v \in W_2)$

$\Rightarrow a=b=c=d=0 \Rightarrow v=0$. $\therefore W_1 \cap W_2 = \{0\}$

(b) Since $\mathbb{R}^4 = W_1 + W_2$, every vector $v \in \mathbb{R}^4$ can be written uniquely as $v = v_1 + v_2$ for $v_1 \in W_1$, $v_2 \in W_2$.

$\therefore$ $T(v)$ is uniquely determined by v.

$\therefore$ T is well-defined.

(c) Note that $T(v) = v_1 + v_2$ for $v_1 \in W_1$, $v_2 \in W_2$ s.t. $v_1 + v_2 = v$,

We have that $T(v) = v$ $\forall\, v \in \mathbb{R}^4$!

$$T(e_1) = e_1 = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot (1,1,1,1)^T + 0 \cdot (0,1,1,-1)^T$$

$$T(e_2) = e_2 = 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot (1,1,1,1)^T + 0 \cdot (0,1,1,-1)^T$$

$$T(e_3) = e_3 = \tfrac{-1}{2} e_1 + (-1) e_2 + \left(\tfrac{1}{2}\right)(1,1,1,1)^T + \left(\tfrac{1}{2}\right)(0,1,1,-1)^T$$

$$T(e_4) = e_4 = \tfrac{-1}{2} e_1 + 0 \cdot e_2 + \left(\tfrac{1}{2}\right)(1,1,1,1)^T + \left(\tfrac{-1}{2}\right)(0,1,1,-1)^T$$

$$\therefore [T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}.$$

(d) $$\left[T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right]_\gamma = \begin{bmatrix} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - \frac{1}{2} z - \frac{1}{2} w \\ y - z \\ \frac{1}{2} z + \frac{1}{2} w \\ \frac{1}{2} z - \frac{1}{2} w \end{bmatrix}.$$

(e) $\because$ $T(v) = v$ $\forall\, v \in \mathbb{R}^4$. T is linear, 1-1, onto.

$\therefore$ T is an isomorphism.