Homework 9.

1.

(a) $\because V = W_1 \oplus W_2$,

$\therefore T(v)$ is uniquely determined by v.

$\therefore T$ is well-defined.

(b) $\gamma = \{ e_1, e_2, (1,1,1,1)^T, (0,1,1,-1)^T \}$

$T(e_1) = 0$, $T(e_2) = 0$,

$T((1,1,1,1)^T) = (1,1,1,1)^T$. $T((0,1,1,-1)^T) = (0,1,1,-1)^T$

$$\therefore [T]_\gamma^\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) $\because [T^n]_\gamma^\gamma = [T \circ T \circ \cdots \circ T]_\gamma^\gamma = \left([T]_\gamma^\gamma\right)^n$

It suffices to show that $\left([T]_\gamma^\gamma\right)^n = [T]_\gamma^\gamma$.

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \left([T]_\gamma^\gamma\right)^2 = [T]_\gamma^\gamma$. By induction, it is done.

2. $\mathcal{L}(V,W) = \{T : V \longrightarrow W \mid T : \text{linear transformation}\}$

(V1) $\forall\ T_1, T_2, T_3 \in \mathcal{L}(V,W)$, $v \in V$

$$\begin{aligned}(T_1 + (T_2 + T_3))(v) &= T_1(v) + (T_2 + T_3)(v) \\ &= T_1(v) + (T_2(v) + T_3(v)) \\ &= (T_1(v) + T_2(v)) + T_3(v) \\ &= (T_1 + T_2)(v) + T_3(v) \\ &= ((T_1 + T_2) + T_3)(v)\end{aligned}$$

$\therefore\ T_1 + (T_2 + T_3) = (T_1 + T_2) + T_3$.

(V2) $\forall\ T_1, T_2 \in \mathcal{L}(V,W)$, $\forall v \in V$

$$\begin{aligned}(T_1 + T_2)(v) = T_1(v) + T_2(v) &= T_2(v) + T_1(v) \\ &= (T_2 + T_1)(v)\end{aligned}$$

$\therefore\ T_1 + T_2 = T_2 + T_1$.

(V3) Let $\mathbf{0} : V \longrightarrow W$, $\mathbf{0}(v) = 0\ \forall\ v \in V . \Rightarrow \mathbf{0} \in \mathcal{L}(V,W)$

$\forall\ T \in \mathcal{L}(V,W)$, $v \in V$

$$\begin{aligned}(\mathbf{0} + T)(v) &= \mathbf{0}(v) + T(v) = T(v) \\ &= T(v) + \mathbf{0}(v) = (T + \mathbf{0})(v)\end{aligned}$$

$\Rightarrow\ \mathbf{0} + T = T = T + \mathbf{0}$

$\therefore\ \mathbf{0}$ is the zero vector in $\mathcal{L}(V,W)$.

(V4) $\forall\, T \in \mathcal{L}(V,W)$,

let $S : V \longrightarrow W$, $S(v) = -T(v)\ \forall\, v \in V$.

Then $S \in \mathcal{L}(V,W)$ and

$$(T+S)(v) = T(v) + S(v) = T(v) - T(v) = 0 = 0(v)$$

$\therefore\ S = -T$.

(V5) $\forall\, T_1, T_2 \in \mathcal{L}(V,W)$, $a \in \mathbb{R}$, $v \in V$

$$\big(a(T_1+T_2)\big)(v) = a\big((T_1+T_2)(v)\big) = a\big(T_1(v) + T_2(v)\big)$$
$$= aT_1(v) + aT_2(v) = (aT_1 + aT_2)(v)$$

$\therefore\ a(T_1+T_2) = aT_1 + aT_2$.

(V6) $\forall\, T \in \mathcal{L}(V,W)$, $a, b \in \mathbb{R}$, $v \in V$,

$$\big((a+b)T\big)(v) = (a+b)T(v) = aT(v) + bT(v)$$
$$= (aT+bT)(v)$$

$\therefore\ (a+b)T = aT + bT$.

(V7) $\forall\, T \in \mathcal{L}(V,W)$, $a, b \in \mathbb{R}$, $v \in V$,

$$\big(a(bT)\big)(v) = a\big((bT)(v)\big) = a\big(bT(v)\big)$$
$$= (ab)T(v) = \big((ab)T\big)(v)$$

$\therefore\ a(bT) = (ab)T$.

(V8) $\forall\ T \in \mathcal{L}(V,W)$, $v \in V$,

$$(1\cdot T)(v) = 1\cdot T(v) = T(v)$$

$$\therefore\ 1\cdot T = T.$$

By (V1) ~ (V8), $\mathcal{L}(V,W)$ is a vector space.

3. Suppose S and T are linear dependent, then $\exists\ a \overset{\neq 0}{} \in \mathbb{R}$ s.t. $S = aT$. We want to claim that in this situation, $R(S) = R(T)$. $\left(S = aT \Leftrightarrow \frac{1}{a} S = T \right)$

For $w \in R(S)$, $\exists\ v \in V$ s.t. $S(v) = w$

$$\Rightarrow (aT)(v) = w \Rightarrow aT(v) = w$$

$$\Rightarrow T(av) = w \quad \therefore\ w \in R(T)$$

$$\therefore\ R(S) \subseteq R(T).$$

For $w \in R(T)$, $\exists\ v \in V$ s.t. $T(v) = w$

$$\Rightarrow \left(\frac{1}{a} S\right)(v) = w \Rightarrow \frac{1}{a} S(v) = w$$

$$\Rightarrow S\left(\frac{1}{a} v\right) = w \quad \therefore\ w \in R(S)$$

$$\therefore\ R(T) \subseteq R(S). \qquad \text{Thus } R(T) = R(S).$$

But T and S are nonzero $\Rightarrow R(T) = R(S) \neq \{0\}$

$\therefore\ R(T) \cap R(S) \neq \{0\}$ $\rightarrow\!\leftarrow$ $\quad \therefore$ linear indep.

4.

$$A=\begin{bmatrix}1&2&3\\1&0&1\\1&-1&1\end{bmatrix}\xrightarrow{\times(-2)}B=\begin{bmatrix}1&0&3\\1&-2&1\\1&-3&1\end{bmatrix}\xrightarrow{\times(-3)}\begin{bmatrix}1&0&0\\1&-2&-2\\1&-3&-2\end{bmatrix}$$

$$\xrightarrow{\times\left(\frac{-1}{2}\right)}\begin{bmatrix}1&0&0\\1&1&-2\\1&3/2&-2\end{bmatrix}\xrightarrow{\times 2}\begin{bmatrix}1&0&0\\1&1&0\\1&3/2&1\end{bmatrix}\xrightarrow{\times(-1)}\begin{bmatrix}1&0&0\\0&1&0\\1&3/2&1\end{bmatrix}$$

$$\xrightarrow{\times(-3/2)}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\xrightarrow{\times(-1)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

$$A\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{3}{2}&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}=I_3.$$

5. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$

be upper triangular matrices, that is, $a_{ij} = b_{ij} = 0 \ \forall i > j$.

We consider the linear transformations

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad T(v) = Av,$$

$$S : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad S(v) = Av.$$

Then let β be the standard basis, we have that $[T]_\beta = A$ and $[S]_\beta = B$.

Note that $AB = [T]_\beta \cdot [S]_\beta = [T \circ S]_\beta$.

$$T(e_j) = \sum_{i=1}^{n} a_{ij} e_i = \sum_{i=1}^{j} a_{ij} e_i \quad (\because a_{ij} = 0 \ \forall i > j)$$

$$S(e_j) = \sum_{i=1}^{n} b_{ij} e_i = \sum_{i=1}^{j} b_{ij} e_i \quad (\because b_{ij} = 0 \ \forall i > j)$$

$$\Rightarrow (T \circ S)(e_j) = T(S(e_j)) = T\left(\sum_{i=1}^{j} b_{ij} e_i\right)$$

$$= \sum_{i=1}^{j} b_{ij} T(e_i) = \underbrace{\sum_{i=1}^{j} b_{ij} \left(\sum_{k=1}^{i} a_{ki} e_i\right)}$$

$$= \sum_{i=1}^{j} c_{ij} e_i \quad \text{for some } c_{ij} \in \mathbb{R}$$

$\therefore [T \circ S]_\beta$ is upper triangular.

6.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 4 & 0 & 1 \\ 6 & -1 & 4 \end{bmatrix} \times(-2) \longrightarrow \begin{bmatrix} 2 & 2 & 4 \\ 0 & -4 & -7 \\ 6 & -1 & 4 \end{bmatrix} \times(-3)$$

$$\longrightarrow \begin{bmatrix} 2 & 2 & 4 \\ 0 & -4 & -7 \\ 0 & -7 & -8 \end{bmatrix} \times\left(\frac{-7}{4}\right) \longrightarrow \begin{bmatrix} 2 & 2 & 4 \\ 0 & -4 & -7 \\ 0 & 0 & \frac{17}{4} \end{bmatrix}.$$