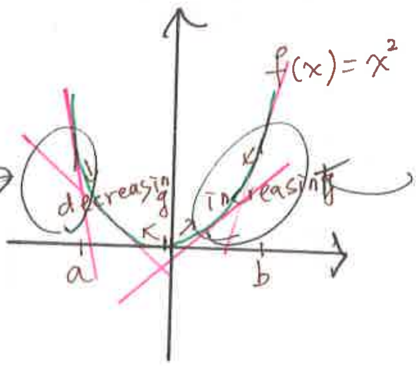
* Increasing and Decreasing Functions. (Monotonic Functions)

Def'n A function $f: I \to \mathbb{R}$ is
(I: interval)

- increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



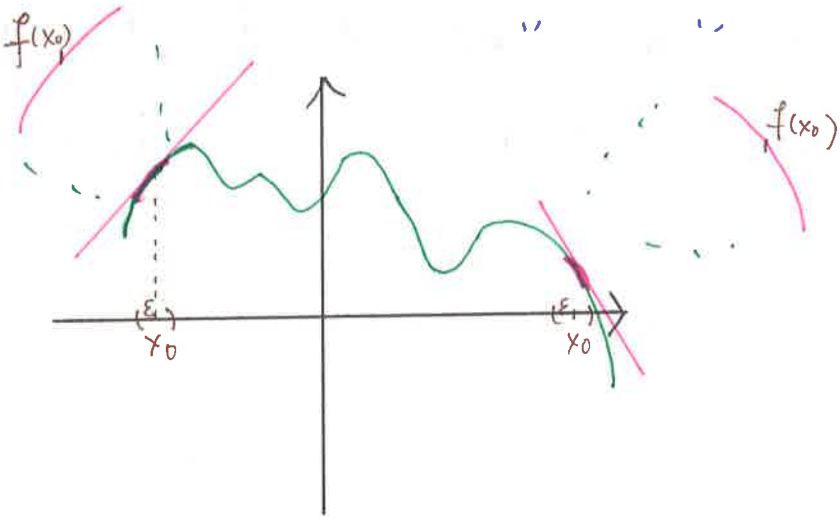
tangent lines of negative slopes (i.e. negative derivatives)

tangent lines of positive slopes (ie positive derivatives)

Verify the above facts more carefully. :

If $f'(x_0) > 0$, then there is $\varepsilon > 0$, s.t. f is increasing on $(x_0 - \varepsilon, x_0 + \varepsilon)$, ie. $f(x_0 - \varepsilon') < f(x_0) < f(x_0 + \varepsilon')$ for all $0 < \varepsilon' < \varepsilon$ .. ⊛

If $f'(x_0) < 0$, " " " " .. decreasing "
" " $f(x_0 - \varepsilon') > f(x_0) > f(x_0 + \varepsilon')$ .. ⊛⊛
" " "



indeed,

$$\frac{1}{\varepsilon}\left[f(x_0+\varepsilon)-f(x_0)\right] \to f'(x_0)$$

as $\varepsilon \to 0$

$$\frac{-1}{\varepsilon}\left[f(x_0-\varepsilon)-f(x_0)\right] \to f'(x_0)$$

$\therefore$ if $f'(x_0) > 0 \Rightarrow \frac{1}{\varepsilon}\left[f(x_0+\varepsilon)-f(x_0)\right] > 0$

$\Rightarrow f(x_0+\varepsilon) - f(x_0) > 0$

AND $\Rightarrow -\frac{1}{\varepsilon}\left[f(x_0-\varepsilon)-f(x_0)\right] > 0$

$\Rightarrow f(x_0-\varepsilon) - f(x_0) < 0$

are true eventually (ie ε small enough)

this shows $\circledast$

(similarly for $\circledast\circledast$)

$\therefore$ if $f' >$ (or $<$) 0 on I, subdividing I into small enough intervals (finitely many) we conclude.

Thm. For $I = [a,b]$ and a function diff. on I,

if $f' > 0$ on $(a,b) \Rightarrow f$ increasing on I

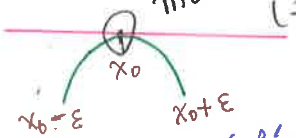
$f' < 0$ " $(a,b) \Rightarrow$ " decreasing " "

What if $f'(x_0) = 0$?

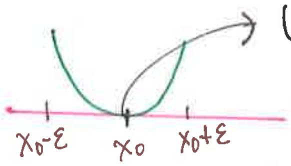
$\underbrace{f'(x_0)=0}$ slope of tangent line to $y = f(x)$ is horizontal.

If f is neither increasing nor decreasing near x_0, there are two possibilities:

local maximum (f' goes from + to −)



f goes up ($f' > 0$) and then down ($f' < 0$)



local minimum (f' goes from + to −)

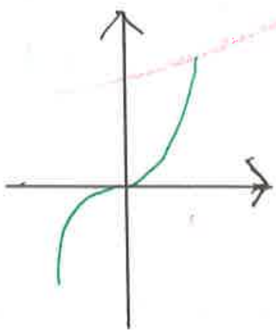
f goes down ($f' < 0$) and then up ($f' > 0$)

Since f' is continuous (CHECK) $\Rightarrow$ $f'(x_0) = 0$.

Thm// If f diff., has local max & min at x_0 $\Rightarrow$ $f'(x_0) = 0$.

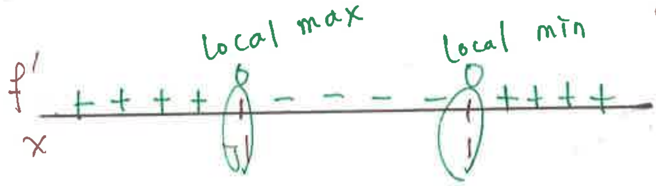
But $f'(x_0) = 0$ doesn't tell anything!

eg// $f(x) = x^3$ & $f'(0) = 0$ But f is increasing at $x=0$



eg:: $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$



positivity table

"critical points" where $f'=0$

$\therefore$ f decreasing on $(-1, 1)$
increasing " $(-\infty, -1)$ & $(1, \infty)$

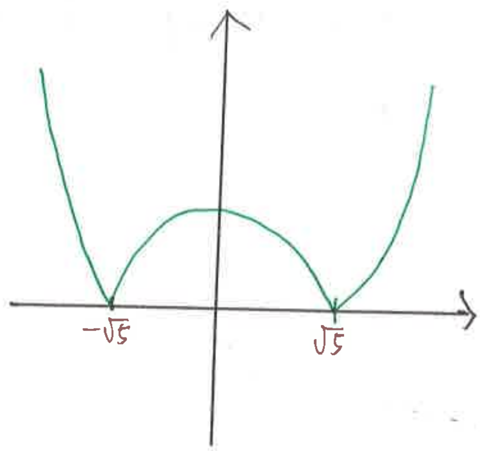
eg:: $f(x) = x^3 + 2x$

$f'(x) = 3x^2 + 2 > 0$ for all x

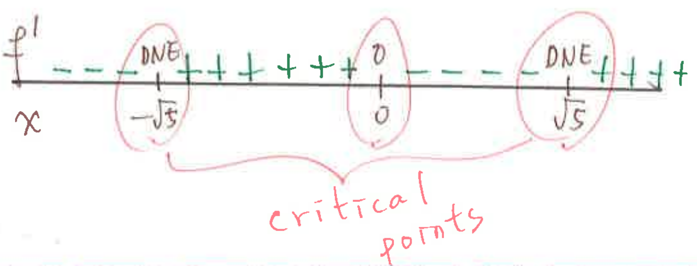
$\Rightarrow$ f is increasing on all $\mathbb{R}$
($\therefore$ no local max & min)

eg:: $f(x) = |x^2 - 5| = \cancel{|(x+\sqrt{5})(x-\sqrt{5})| = \begin{cases} -(x+\sqrt{5})(x-\sqrt{5}) ; & x \in (-\sqrt{5}, \sqrt{5}) \\ (x+\sqrt{5})(x-\sqrt{5}) ; & x \in (-\infty, -\sqrt{5}) \\ & x \in (\sqrt{5}, \infty) \end{cases}}$

$$= \begin{cases} x^2 - 5 & ; \ x \notin [-\sqrt{5}, \sqrt{5}] \\ 5 - x^2 & ; \ x \ [-\sqrt{5}, \sqrt{5}] \end{cases}$$

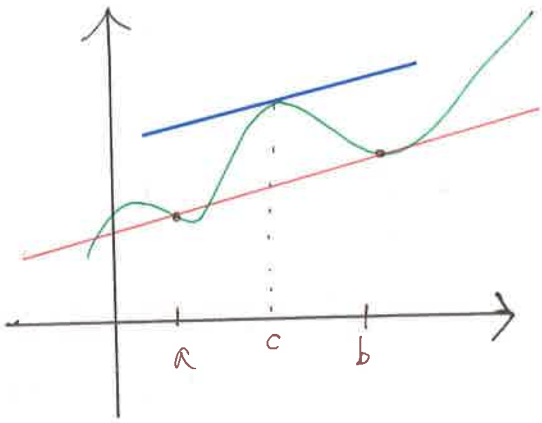


$$f'(x) = \begin{cases} 2x & ; \ x \notin (-\sqrt{5}, \sqrt{5}) \\ -2x & ; \ x \in (-\sqrt{5}, \sqrt{5}) \end{cases}$$



V. Applications of Differentiations

Intuitive idea



f diff. on $[a,b]$ then there is $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

average velocity from time $x=a$ to $x=b$

at some time b/w a & b, the particle will move as fast as the average velocity, otherwise velocity is always lower or higher than average (not possible!)

* Take as granted: Continuous function attains a max & min on a closed interval.

Also clear:

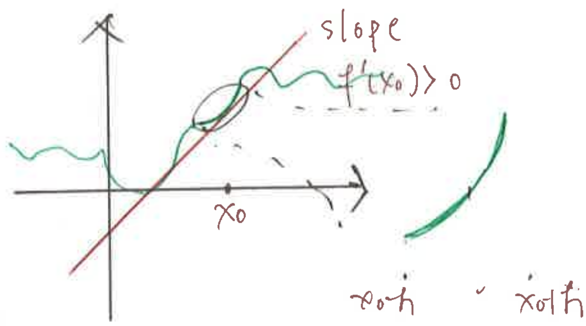
f diff. at x_0, if

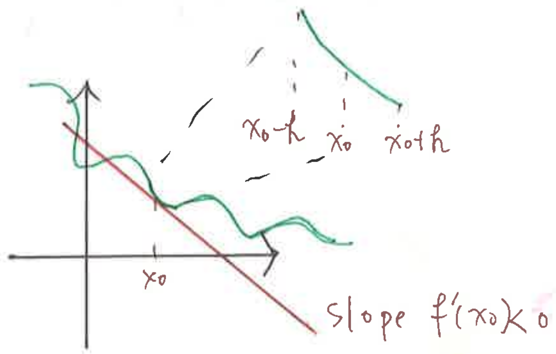
$f'(x_0)>0$, then $f(x_0-h) \le f(x_0) \le f(x_0+h)$ $\forall h$ small

i.e. f is increasing on (x_0-h, x_0+h)

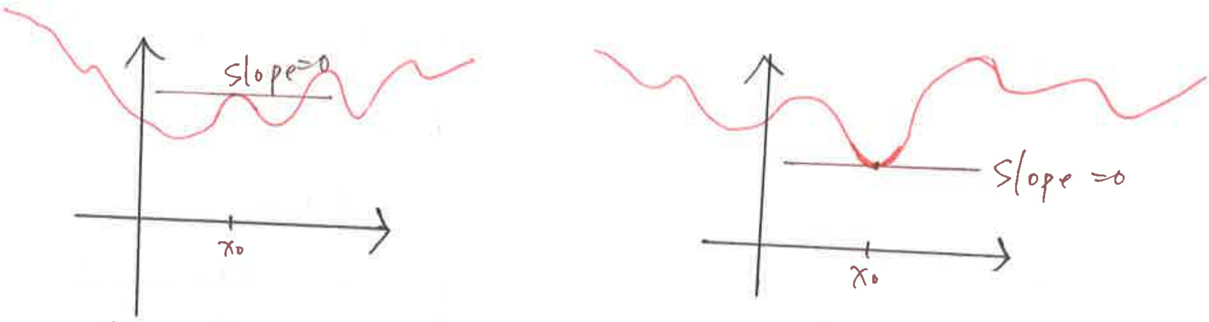
$f'(x_0)<0$, then $f(x_0-h) \geq f(x_0) \geq f(x_0+h)$ $\forall h$ small

i.e. f is decreasing on (x_0-h, x_0+h)





$\therefore$ If f is not increasing nor decreasing near $x_0 \Rightarrow f'(x_0)=0$



$f(x_0)$ is a local maximum $f(x_0) \geq f(x_0+h)$ $\forall h$

$f(x_0)$ is a local minimum $f(x_0) \leq f(x_0+h)$ $\forall h$

(But the converse is not true!) eg. $f(x)=x^3$.

Thm. (Rolle's Theorem)

f continuous on $[a,b]$ and diff. on (a,b) and $f(a)=f(b)=0 \Rightarrow \exists\, c \in (a,b)$ s.t. $f'(c)=0$

Pf. if f constant $\Rightarrow f'=0$ on (a,b) & done.

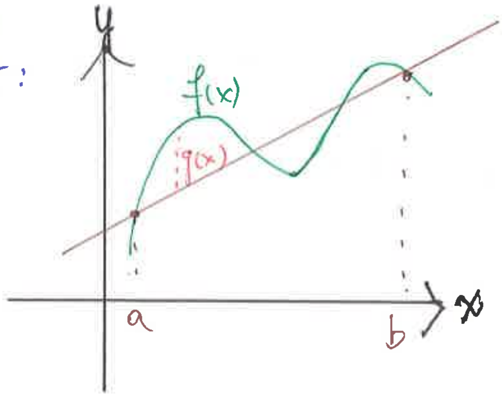
if not, by a classical thm., cont function attains max and min on closed interval.

say $f(c)$ is max $\Rightarrow f(c)>0 \Rightarrow c \neq a$ or b.

& since $f(c)$ is max $\Rightarrow$ it is a local max

$\Rightarrow f'(c)=0$ //

Proof of MVT: $y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a) = h(x)$



consider $g(x) = f(x) - h(x)$

clearly $g(a) = g(b) = 0$

$\therefore$ there is c in (a,b) s.t. $g'(c)=0$

But $g'(x) = f'(x) - h'(x)$

$= f'(x) - \frac{f(b)-f(a)}{b-a}$

$\Rightarrow$ $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$ ※.

eg// There is no diff. function w/ $f(0)=2$, $f(2)=5$ and $f'(x) \leq 1$ for all x in $(0,2)$.

Since $\frac{f(2)-f(0)}{2-0} = \frac{5-2}{2} = \frac{3}{2} > 1$. & by MVT there must be some c in $(0,2)$ s.t. $f'(c) = \frac{3}{2}$.

eg// $f(x) = x^3 + 9x^2 + 33x - 8$ has exactly one root

pf// $f(x)$ has a real root. (By Intermediate value thm)

but if there is more than one, say $x_1 < x_2$ s.t. $f(x_1) = f(x_2) = 0 \Rightarrow f'(c) = 0$ for some c in (x_1, x_2). But $f'(x) = 3x^2 + 18x + 33$ has no real root

⟶

Since $18^2 - 4 \cdot 3 \cdot 33 < 0$ $\rightarrow\leftarrow$.

$\therefore$ ONLY ONE REAL ROOT.

* Extrema (Extreme Values) suppose $D' = \{x \in A \mid f'(x) \text{ DNE}\}$ is ... ⑨

$f: A \to \mathbb{R}$ is said to have an ...

maximum at $c \in A$ if $f(c) \geq f(x)$ for all x in A

minimum at $c \in A$ if $f(c) \leq f(x)$ for all x in A

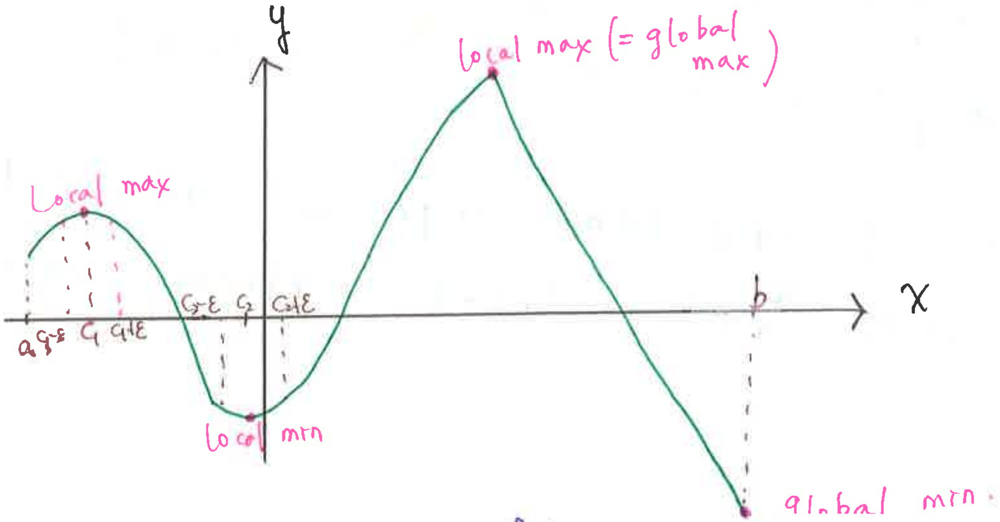
(they're also called "global max/min." on the set A)

Assumption: $f' = 0$ OR DNE only finitely many times.

We look for max/min on A by listing all the "suspects" and compare the values of f at them.

1. Local extreme values.

f has local max (min) at c if there is $\varepsilon > 0$ so that $f(c)$ is global max (min) of $(c-\varepsilon, c+\varepsilon)$.



$A = [a, b]$

10/29 Ends

How to find/classify local extrema?

if f differentiable, then $x = c$ local max/min $\Rightarrow f'(c) = 0$

$\therefore$ suspect list: $\underbrace{\{c \mid f'(c) = 0\} \cup \{c \mid f'(c) \text{ DNE}\}}_{\text{critical points}}$

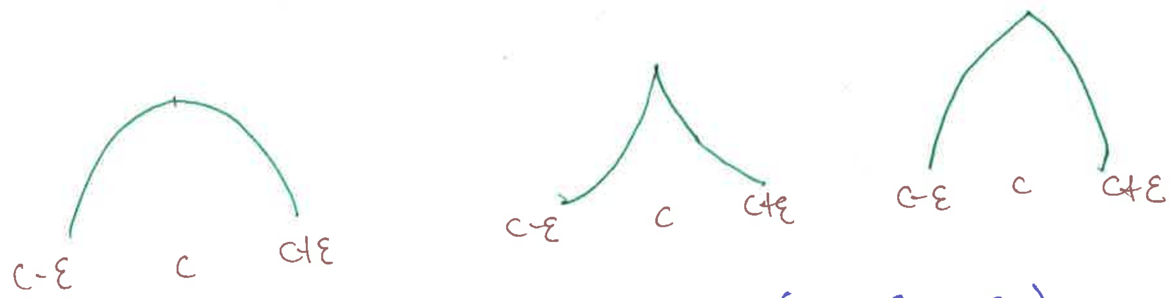
$\{\text{Local max/min}\} \subset \{\text{critical points}\}$

Next step: capture local max/min from suspects

Recall: $f(c)$ is local max (min) if $f(c)$ is max (min) on $(c-\varepsilon, c+\varepsilon)$, i.e. $f(x) \le f(c)$ ($\geq$) for all x in $(c-\varepsilon; c+\varepsilon)$, for some $\varepsilon>0$. Pick ε small enough so that f' don't change sign on $(c-\varepsilon, c)$ & $(c, c+\varepsilon)$

(i.e. either increasing/decreasing on either half)

$\therefore$ if $f(c)$ is local max



$f(c) \geq f(x)$ For all x in $(c-\varepsilon, c)$

$\Leftrightarrow$ f increasing on $(c-\varepsilon, c)$

$\Leftrightarrow$ $f' > 0$ " "

$f(c) \geq f(x)$ - For all x in $(c, c+\varepsilon)$

$\Leftrightarrow$ f decreasing on $(c, c+\varepsilon)$

$\Leftrightarrow$ $f' < 0$ " "

$\therefore$ at local max, f' switches from $+$ to $-$ across c

Similarly,

at local min, f' " " $-$ to $+$ " "

This is called First derivative test

☆☆☆ Do Example on 1st p4

eg. $f(x) = \frac{1}{2}x^2 + \frac{1}{x}$

$$f'(x) = x - \frac{1}{x^2} = \frac{x^3 - 1}{x^2}$$

critical points: $f'(x) = 0$ at $x = 1$

f' DNE at $x = 0$

as $f \to \infty$ as $x \to \pm\infty$

No local max

Local min $f(1) = \frac{3}{2}$

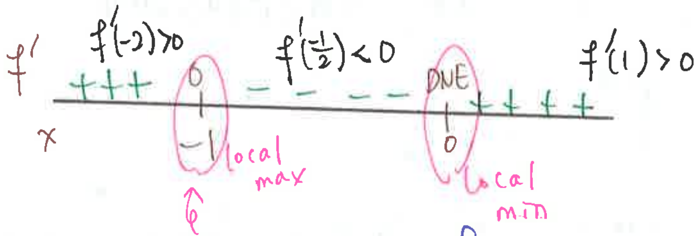
Note: if all zeros of f' & discontinuities are listed, to determine $f' > 0$ OR $f' < 0$ b/w zeros, we only need to plug in one point

eg. $f(x) = 2x^{\frac{5}{3}} + 5x^{\frac{2}{3}}$

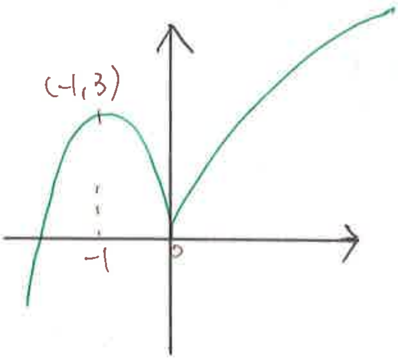
$$f'(x) = \frac{10}{3}x^{-\frac{1}{3}}(x+1) = \frac{10(x+1)}{\sqrt[3]{x}} \quad ; \ x \neq 0$$

critical pt:

$x = 0$ & $x = -1$

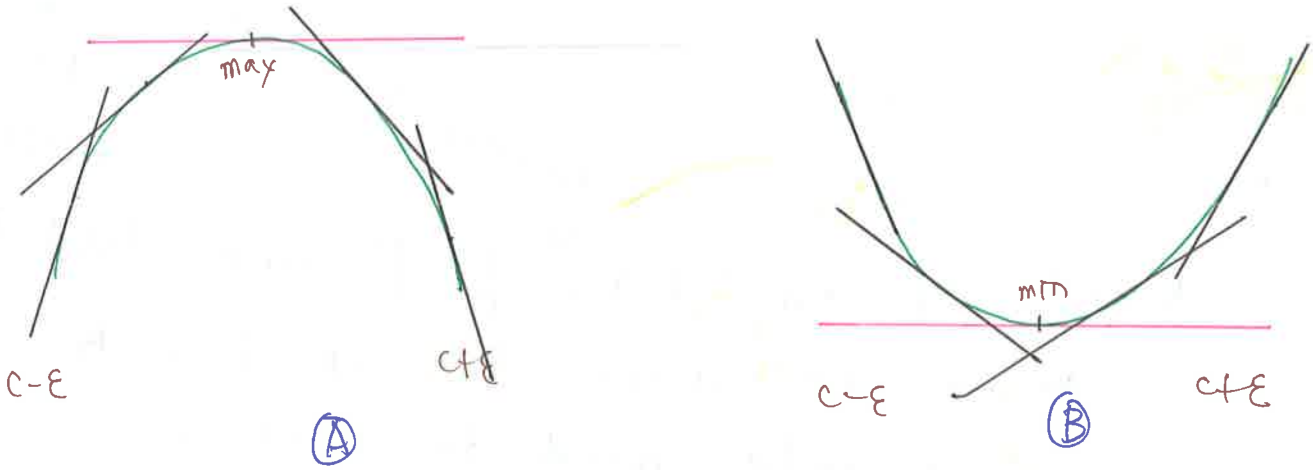


$f(-1) = 3$ $f(0) = 0$



For f C^2 function, we don't have suspect like

only



Observe, slopes of tangent lines in A (f') is increasing ($+$big $\to$ $+$small $\to 0 \to$ $-$small $\to$ $-$big)

ie f' is an decreasing function on $(c-\varepsilon, c+\varepsilon)$

$\Rightarrow$ $f''<0$

Similarly, $f''>0$ on $(c-\varepsilon, c+\varepsilon)$ in B

This method is called the

Second derivative test.

eg. $f(x) = x^3 + 2x^2 + x$ (C^∞ function)

$f'(x) = 3x^2 + 4x + 1 = (3x+1)(x+1)$

$f''(x) = 6x + 4$

$f' = 0$ at $\underbrace{x = -\frac{1}{3} \ \& \ -1}_{\text{crit. points}}$.

$f''(-\frac{1}{3}) = -2 + 4 = 2 > 0$ $\therefore$ $f(-\frac{1}{3}) = -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} = \frac{-1+6-9}{27}$

$f''(-1) = -6 + 4 < 0$ $= \frac{-4}{27}$

is local min.

$f(-1) = -1 + 2 - 1 = 0$

is local max.

//

Continue the search toward global (absolute) extreme values ... (Principle: Find all suspects & compare values of f at each of them)

Suspects for absolute extremas:

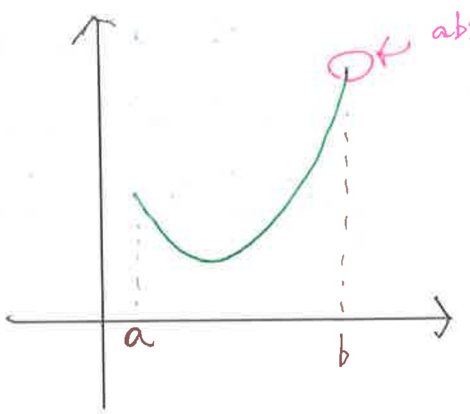
certainly, local extremas are possible absolute extremas. what else?

consider $f: [a,b] \to \mathbb{R}$ (Recall, there has to be max/min if f continuous)

We know all the suspects in (a,b). are $\{c \mid f'(c)=0\}$ & $\underbrace{\{c \mid f'(c) \text{ DNE}\}}_{}$ $= \text{crit}(f)$

The only points we're not sure are $x=a$ & b.

$\therefore$ add them to suspect list and we've included all possibilities.



eg.) $f: [-1,3] \to \mathbb{R}$

$f(x) = 1 + 4x^2 - \frac{1}{2}x^4$ (f' exists everywhere)

$f'(x) = 8x - 2x^3 = 2x(2-x)(2+x) = 0$ when $x = 0, 2, \cancel{-2}$

not in $[-1,3]$

critical pts: $\{0, 2\}$.

$\therefore$ suspect list $\{0, 2, -1, 3\}$

$f(0) = 1$, $f(2) = 9$, $f(-1) = \frac{9}{2}$, $f(3) = -\frac{7}{2}$

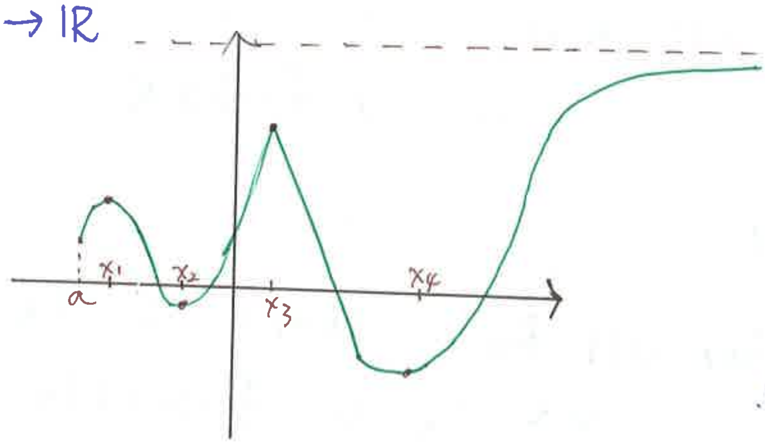
abs. max   abs. min

What about unbounded domain?

$f: A \to \mathbb{R}$ where A is not contained in any interval.

$\therefore$ either $x \to \infty$, or $-\infty$ (or both)

$f: [a, \infty) \to \mathbb{R}$



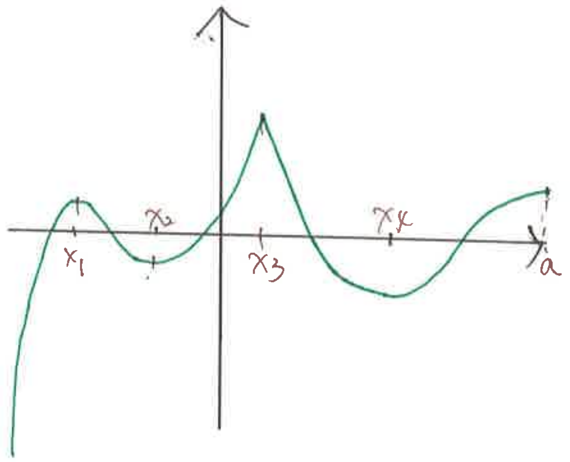
$\lim_{x \to +\infty} f(x) = M$

$> f(x_1), f(x_2), f(x_3)$

$f(x_4), f(a)$

$\therefore$ No abs. max.

$f: (-\infty, a] \to \mathbb{R}$



$\lim_{x \to -\infty} f(x) = -\infty$

$< f(x_1), \dots, f(x_4), f(a)$

$\therefore$ No abs. min

Conclusion:

if $\lim_{x \to +\infty} f(x)$ (or $\lim_{x \to -\infty} f(x)$) $= M$ (maybe ∞) so that

$M < f(x)$ for all crit. pts $x \Rightarrow$ no abs. min

$M > f(x)$ " " " $x \Rightarrow$ no abs. max.

($\therefore$ if $M = +\infty \Rightarrow$ no abs max

$-\infty \Rightarrow$ no abs. min

Rmk.

$$\lim_{x \to \infty} f(x) = M$$

$\Leftrightarrow$ $\begin{cases} M < \infty : & \text{For all } \varepsilon > 0, \text{ there is } L_K > 0 \text{ s.t.} \\ & x > L_K \Rightarrow |f(x) - M| < \varepsilon \\ M = \infty : & \text{For all } K > 0, \text{ " " " "} \\ & \text{" " } \Rightarrow f(x) > K \end{cases}$

$$\lim_{x \to -\infty} f(x) = M$$

$\Leftrightarrow$ $\begin{cases} M < \infty : & \text{For all } \varepsilon > 0, \text{ there is } L_K > 0 \text{ s.t.} \\ & x < -L_K \Rightarrow |f(x) - M| < \varepsilon . \\ M = \infty : & \text{For all } K > 0, \text{ " " " "} \\ & \text{" " } \Rightarrow f(x) > K \end{cases}$

Conclusion for extrema hunting: of $f: A \to \mathbb{R}$.

1. Find critical points

$$\mathrm{crit}(f) = \{x \in A \mid f'(x) = 0 \text{ OR } f'(x) \text{ DNE}\}$$

Sometimes tricky to find

$$= \{x_1, \dots, x_n\}$$

$n < \infty$, initial assumption

2. Suspect Lists = crit(f) ∪ {end points of each interval}

3. if A unbounded, find $\underbrace{\lim_{x \to \infty} f(x)}_{= L_1}$ & $\underbrace{\lim_{x \to -\infty} f(x)}_{= L_2}$ (or both)

4. compare $f(x)$ for each $x \in \mathrm{crit}(f)$

let $M = \max\{f(x_1), \dots, f(x_n)\} \overset{\text{say}}{=} f(x_i)$

$m = \min\{f(x_1), \dots, f(x_n)\} \overset{\text{say}}{=} f(x_j)$

$M > L_1$ AND L_2 ?
- YES → $f(x_i) = M$ abs. max
- NO → No abs. max

$m < L_1$ AND L_2 ?
- YES → $f(x_j) = m$ abs min
- No → No abs min.

If asked to classify critical points, use 1st derivative test (or second derivative test when f is C^2.)

eg₁₁ $f(x) = x^3 - 2x^2 + 7x \quad : (-\infty, \infty) \to \mathbb{R}$

$\lim_{x\to+\infty} f(x) = +\infty$ & $\lim_{x\to-\infty} f(x) = -\infty$

no abs. max

no abs. min

eg₁₁ $f(x) = \begin{cases} -x^2 ; & 0 \le x < 1 \\ -2x ; & 1 \le x < 2 \\ -\frac{1}{2}x^2 ; & 2 \le x < \infty \end{cases} \quad : [0, \infty) \to \mathbb{R}$

$\lim_{x\to\infty} f(x) = -\infty \Rightarrow$ no abs. min.

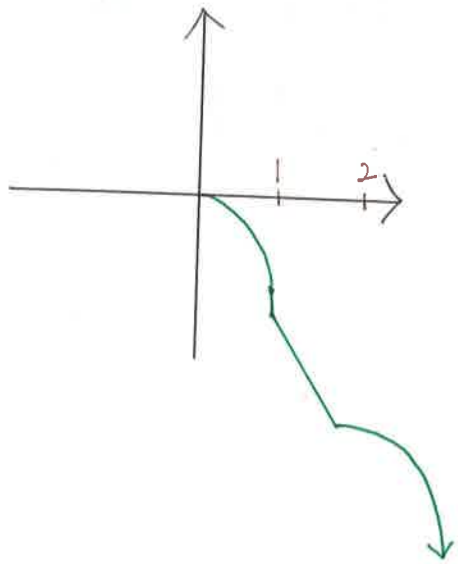
critical pts : $f'(x) = \begin{cases} -2x ; & 0 < x < 1 \\ -2 ; & 1 < x < 2 \\ -x ; & 2 < x < \infty \end{cases}$

$f' \neq 0$ anywhere. f' exists everywhere.

$\therefore$ only need to look for abs. max from endpoints $\{0, 1, 2\}$

$f(0) = 0$, $f(1) = -2$, $f(2) = -4$

abs. max.



eg 11 $f(x) = \sin^2 x - \sqrt{3}\cos x \quad ; \quad x \in [0, \pi]$

Find extreme values & classify crit(f).

$f'(x) = 2\sin x\cos x + \sqrt{3}\sin x = \sin x(2\cos x + \sqrt{3})$ (exists everywhere)

$f' = 0$ when $\sin x = 0$ OR $\cos x = -\frac{\sqrt{3}}{2}$

$\therefore \text{crit}(f) = \{0, \pi, \frac{5\pi}{6}\}$

Classification:

f'	0	$+ + + + + + +$	0	$-$	0
x	0		$\frac{5\pi}{6}$		π

$f'(\frac{\pi}{2}) = 1(0+\sqrt{3}) > 0$

take x very close to π (but <) $\sin x > 0$ & $\cos x \to -1$ $\Rightarrow f' < 0$

$\therefore f(\frac{5\pi}{6}) = \frac{1}{4} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = \frac{10}{4}$ is local max.

(may also try $f'' = 2\cos(2x) + \sqrt{3}\cos x$

$f''(\frac{5\pi}{6}) = 2\cos(\frac{5\pi}{3}) + \sqrt{3}\cos\frac{5\pi}{6} < 0 \Rightarrow$ local max)

Extrema: suspect list $= \{0, \frac{5\pi}{6}, \pi\}$

$f(0) = -\sqrt{3}$, $f(\frac{5\pi}{6}) = \frac{10}{4}$, $f(\pi) = \sqrt{3}$

abs. min abs. max

* Concavity - Geometric Meaning of f''

Recall : $f' \leftrightarrow y = f(x)$ going up or down.

What geometric info. does f'' carry ?

Say, $f' \geq 0$, it can increase

like  or 

x_0 x_0

- keep turning Left
- increase faster & faster
- slope of tangent line $(f' \geq 0)$ gets larger & larger (ie. f' increasing) $f'' \geq 0$ (if f is C^2)

→ f "concaves up" near x_0

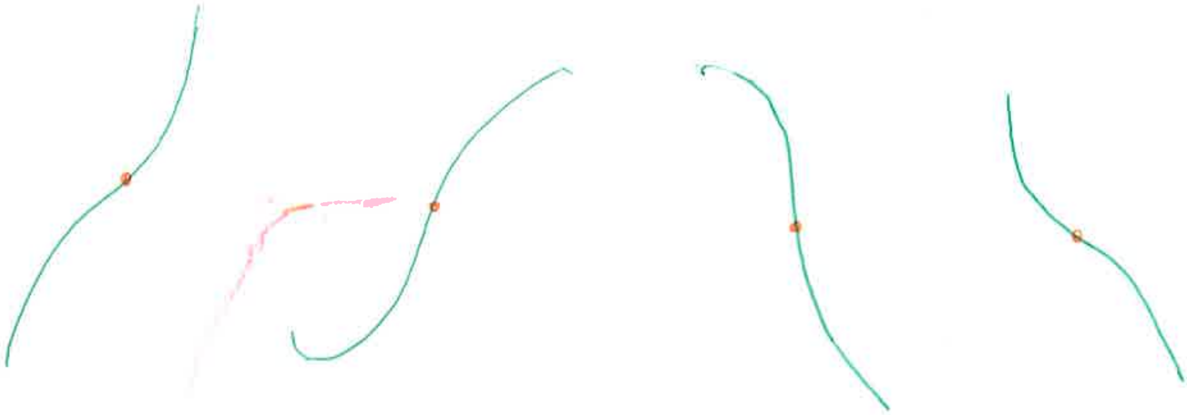
- keep turning right.
- increase slower & slower.
- slope of tangent lines $(f' \geq 0)$ gets smaller & smaller (if f' decreasing) $f'' \leq 0$ (if f is C^2)

→ f concaves down near x_0.

Def^n (Point of Inflection)

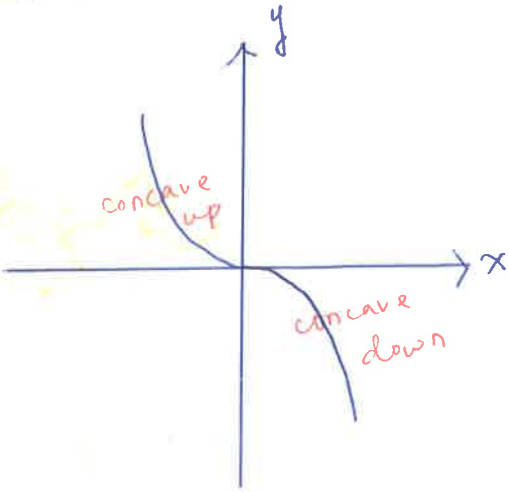
x_0 is called a point of inflection if concavities of f on $(x_0-\varepsilon, x_0)$ and $(x_0, x_0+\varepsilon)$ are opposite.

If f is C^2, x_0 is point of inflection $\Rightarrow f''(x_0) = 0$.



Other possibilities of point of inflection:

$$f(x) = \begin{cases} x^2 ; & x<0 \\ -x^3 ; & x \geq 0 \end{cases} \Rightarrow f''(0) \text{ DNE}$$



$$f'(x) = \begin{cases} 2x ; & x<0 \\ -3x ; & x \geq 0 \end{cases}$$

increasing

decreasing

$$f''(x) = \begin{cases} 2>0 & x>0 \\ \underset{<0}{-6x} ; & x>0 \end{cases}$$

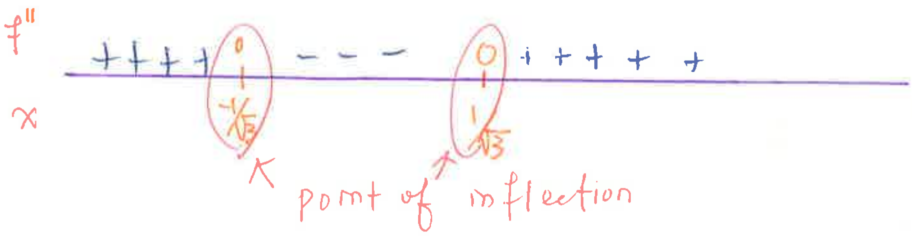
But $f''(0)$ DNE

$\therefore$ $\boxed{\text{Suspect for Point of Inflection } = \{c \mid f''(c)=0\} \cup \{c \mid f''(c) \text{ DNE}\}}$

eg.

$$f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$$

$$f'(x) = x^3 - x \quad , \quad f''(x) = 3x^2 - 1$$



* Curve Sketching

Plot down points (x, y) where $y = f(x)$ with x in the domain.

- Precisely understood with lines (just plot two points and connect)
- "Somehow" understood with $y = (x-a)^2 + b$.
- No systematic ways for more general functions

→ Gather geometric info of $y = f(x)$, and reflect all of them on the graph as much as we can.

eg.. $f(x) = x^4 - 4x^3 + 4x^2$, $-1 \le x < 5$

Domain:
all x in $[-1, 5)$

Intercepts:
$f(0) = 0$;
$f(x) = 0$ at $x = 0, 2$
(graph contains $(0,0), (2,0)$)
$f(-1) = 9$, $f(5) = 125$

Monotoncity
$f'(x) = 4x^3 - 12x^2 + 8x = x(4x^2 - 12x + 8)$
$f' = 0$ at $0, 1, 2$

Geometric Insights / General Strategies

- Determine when $\frac{\times\times}{0}$, $\sqrt{<0}$ occur. (or any domain by assumption)
- Plug in a few convinient points, usually $(0, f(0))$ y intercept and $(x_0, 0)$ where $f(x_0) = 0$. x intercept and (end pts, f(end pts))
- whether $y = f(x)$ rises / falls and whether f arrives at peak / bottom .. determine the nature of crit. points

Concavity

$f''(x) = 12x^2 - 24x + 8$

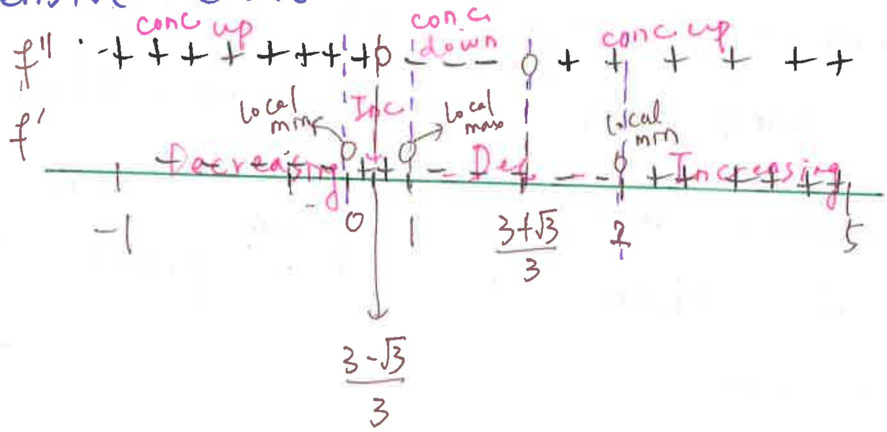
$f'' = 0$ at

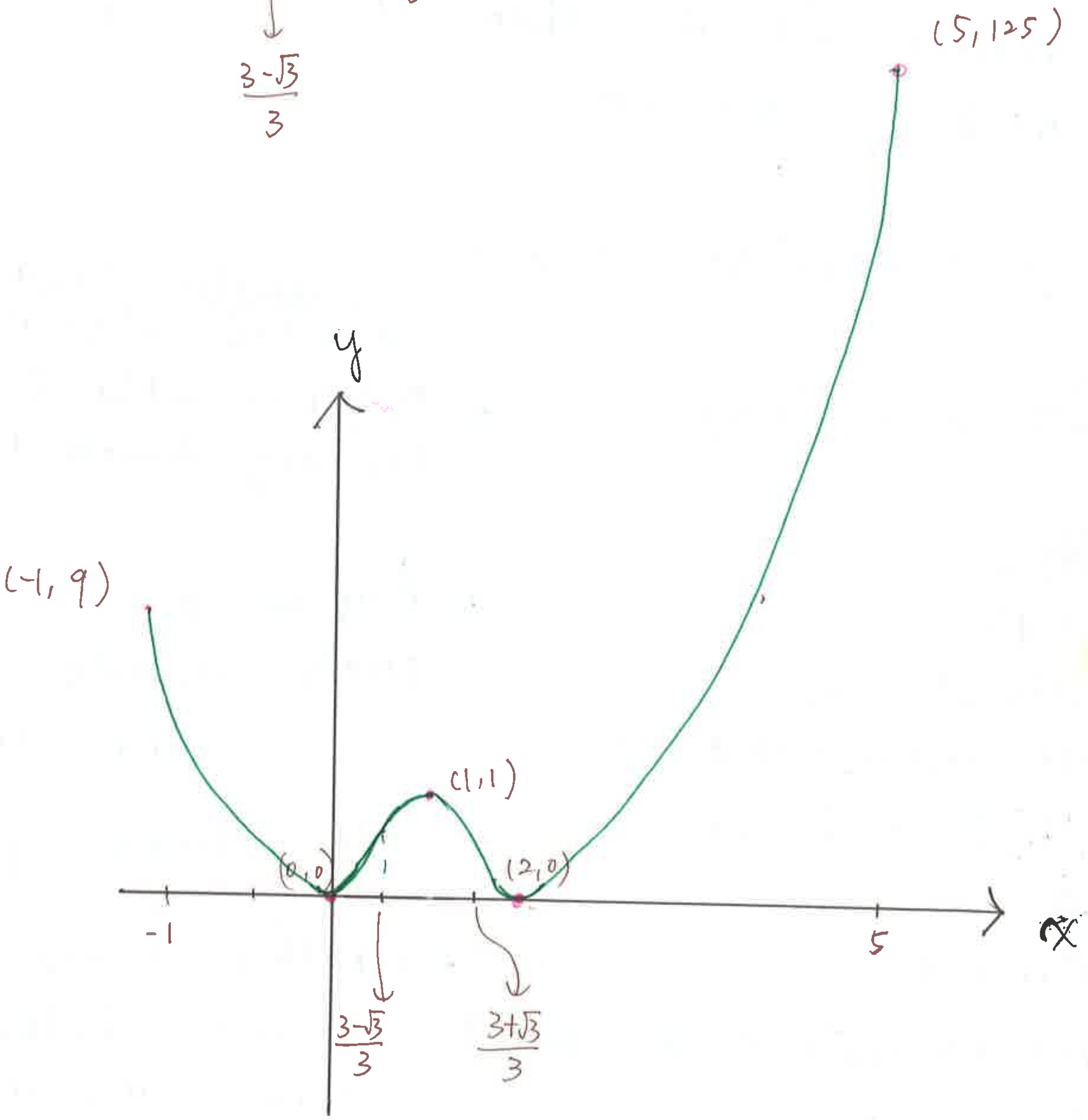
$x = \frac{3 \pm \sqrt{3}}{3}$

- How the curve increases/decreases (turning right/left?)
- determin signs of f'' across point of inflection

can be down together

Comprehensive table :



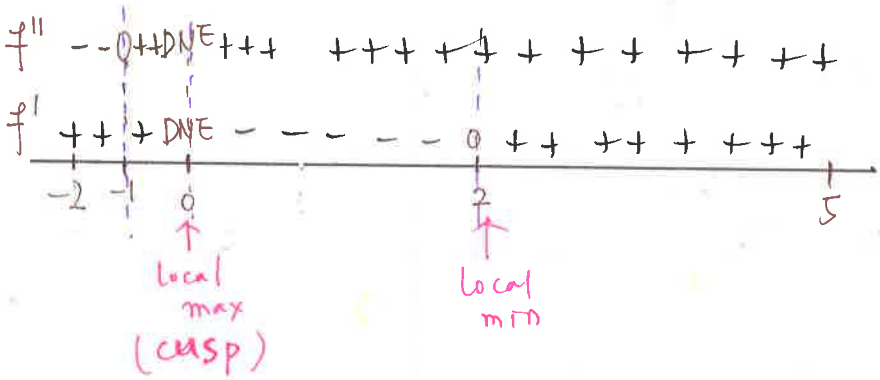


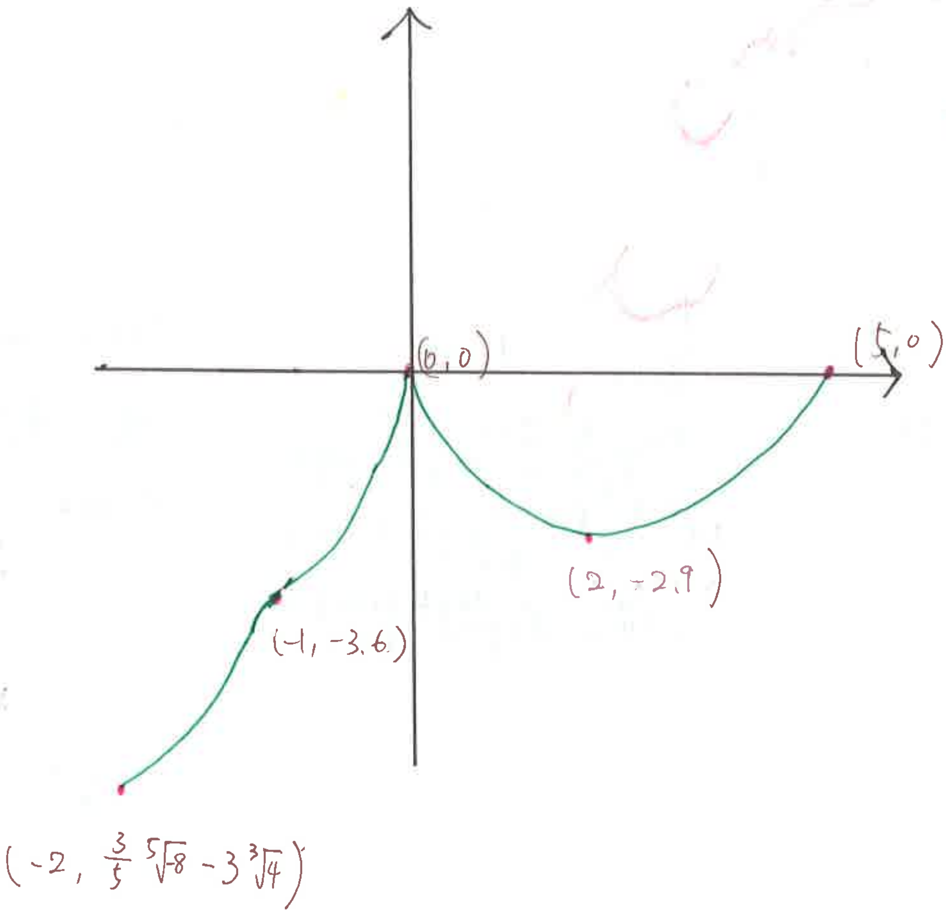
eg 11 $f(x) = \frac{3}{5}x^{\frac{5}{3}} - 3x^{\frac{2}{3}}$; $x \in [-,5]$

$f'(x) = \dfrac{x-2}{x^{1/3}}$; $f' = 0$ at 2 & DNE at 0

$f''(x) = \frac{2}{3}x^{-\frac{3}{4}}(x+1)$; $f'' = 0$ at $x =$ DNE at $x = 0$

$f(0) = 0$ and $f(5) = 0$





Dealing with ∞:

Situations not discussed above:

$\lim_{x\to c} f(x) = \pm\infty$; $\lim_{x\to\pm\infty} f(x) = M$; $\lim_{x\to c} f' = \pm\infty$

(see p.15-16 for definitions)

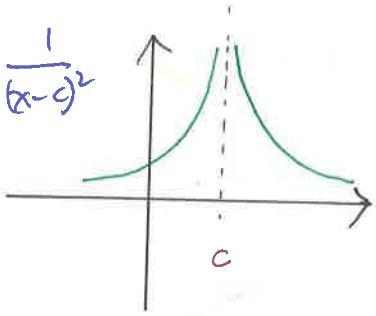
Vertical Asymptoes

Horizontal Asymptoes (if $|M| < \infty$)

Vertical Tangent Lines

A//

$f(x) = \frac{1}{(x-c)^2}$



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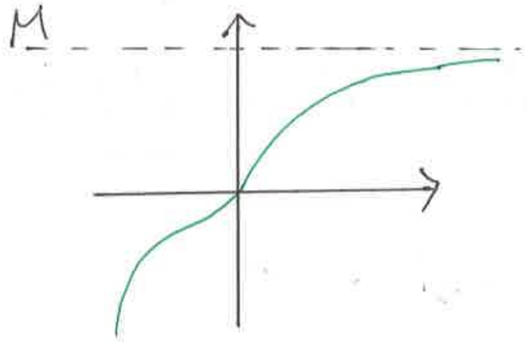

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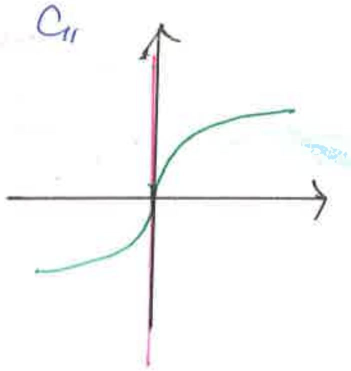
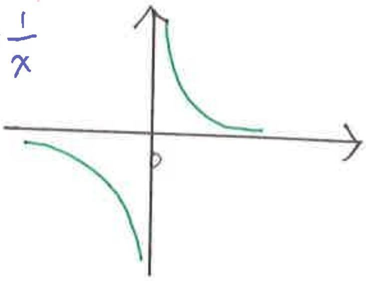


$f(x) = \frac{1}{x}$



Common Ocasions:

① $\frac{xxx}{0}$ at some finite x.

② 0 inside log

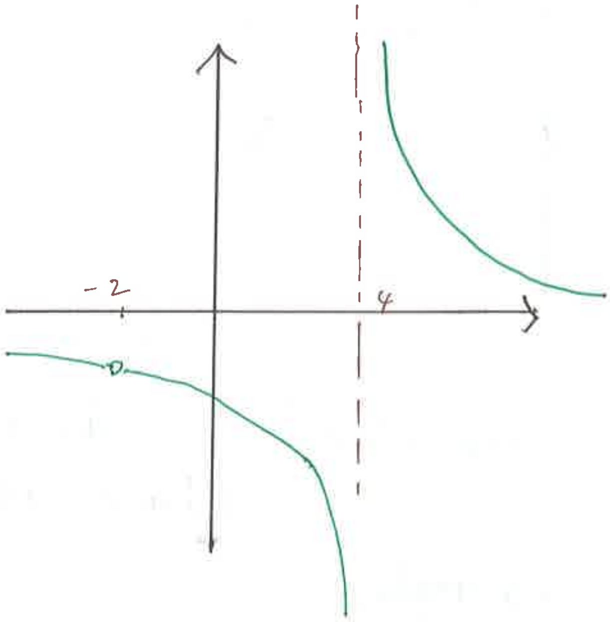
$\frac{P(x)}{q(x)}$ where

$$M \begin{cases} 0 \,; & \text{if } \deg q > \deg P \\ \text{nonzero } \# \,; & \text{if } \deg q = \deg P \\ \infty \,; & \text{if } \deg q > \deg P \end{cases}$$

$f(x)$ w/ f' of type A

eg 11 (A) $f(x) = \frac{3x+6}{x^2-2x-8} = \frac{3(x+2)}{(x+2)(x-4)} \left(\underset{x \neq -2}{\overset{\text{for all}}{=}} \frac{3}{x-4} \right)$

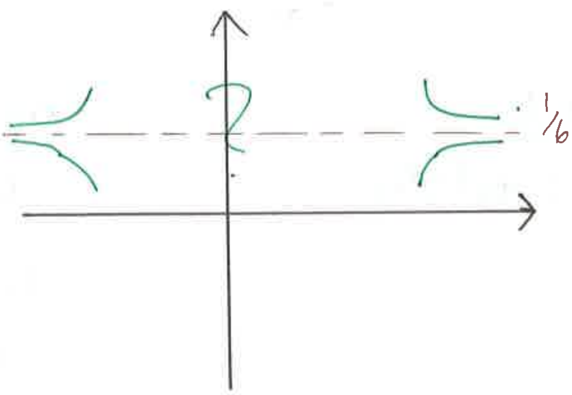
$\lim_{x \to 2} f(x) = \frac{3}{-2}$ $\qquad$ $\lim_{x \to 4^+} f(x) = +\infty$



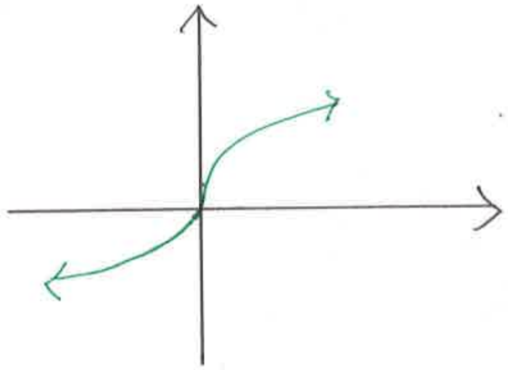
(B) $f(x) = \frac{x^2+4x+3}{6x^2-18x+9}$

$$\lim_{x \to \infty} f(x) = \frac{1 + 4/x + 3/x^2}{6 - 18/x + 9/x^2} = \frac{1}{6}$$

(only need to compute $\lim_{x \to \pm\infty} \frac{\text{leading term}}{\text{leading term}}$)



(c) $f(x) = x^{1/3}$: $\lim_{x \to 0^-} f'(x) = +\infty$ & $\lim_{x \to 0^+} f'(x) = +\infty$



More qualitative properties to observe :
(time saving!)

Properties	examples	Why it matters?
Parity • f is even / odd if $f(-x) = \begin{cases} f(x) \\ -f(x) \end{cases}$	$f(x) = \begin{cases} \sin x \to \text{odd} \\ \cos x \to \text{even} \end{cases}$ $f(x) = \begin{cases} x^3 \to \text{odd} \\ x^{18} \to \text{even} \end{cases}$	only need to graph positive (or negative) half of the domain. eg: $f : [-2, 7] \to \mathbb{R}$ (only need to graph $f : [0, 7]$).
Periodicity • f is periodic w/ period T if $f(x+T) = f(x)$ for all x	trig. functions	only need to graph f on $[0, T]$ (or any $[x, x+T]$) and the rest are just copies of it.

eg$_{11}$ $f(x) = \dfrac{x^2-3}{x^3} : \mathbb{R} \to \mathbb{R}$

Note: $f(-x) = -f(x)$ $\therefore$ only need to sketch $[0,\infty)$

$$f'(x) = \frac{9-x^2}{x^4} \Rightarrow \begin{cases} 0 & \text{at } \pm 3 \\ \text{DNE} & \text{at } 0 \end{cases}$$

$$f''(x) = \frac{2(x^2-18)}{x^5} \Rightarrow \begin{cases} 0 & \text{at } \pm\sqrt{18} \\ \text{DNE} & \text{at } 0 \end{cases}$$

$\lim_{x\to 0^+} f(x) = -\infty$ (vertical asymp.) $f(x)=0$ at $\sqrt{3}$

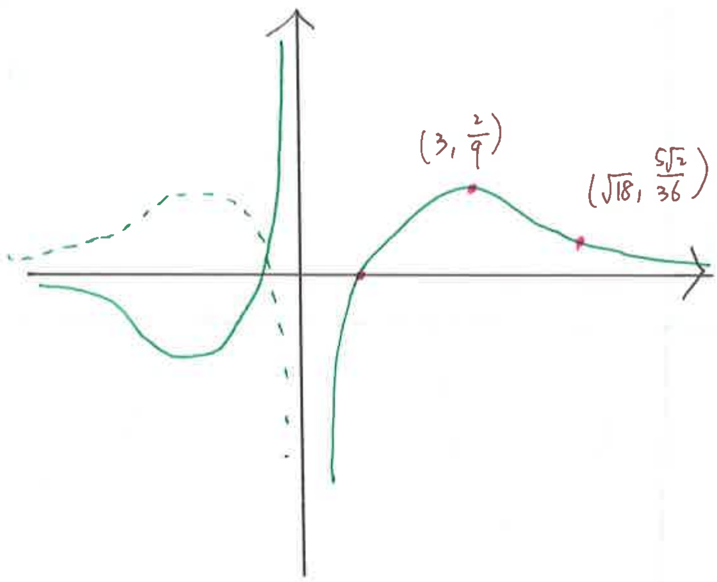
$\lim_{x\to+\infty} f(x) = 0$

f'' DNE − − − − − − − − − 0 + + + +

f' DNE + + + + + + + + + + 0 − − − − − − − − −

0 3 $\sqrt{18}$

$f(3) = \frac{2}{9}$ $f(\sqrt{18}) = \frac{5\sqrt{2}}{36}$



eg4 $f(x) = \sin(2x) - 2\sin x \; : \mathbb{R} \to \mathbb{R}$

f periodic w/ period 2π $\Rightarrow$ only need to sketch f on $[-\pi, \pi]$

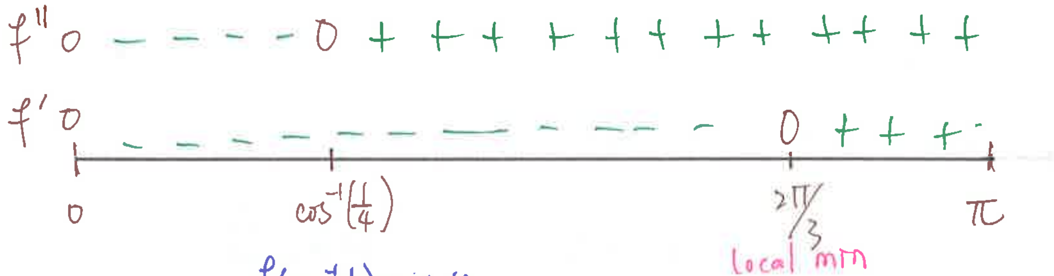
$f(-x) = -f(x)$ $\Rightarrow$ " " " $[0, \pi]$.

$f'(x) = 2(2\cos x + 1)(\cos x - 1)$; $f''(x) = 2\sin x(-4\cos x + 1)$

$\therefore f' = 0$ at $0, \frac{2\pi}{3}$

$f'' = 0$ at $0, \cos^{-1}(\frac{1}{4})$ (≈ 1.3)

No ∞ involved and $\lim_{x \to \pm\infty} f(x)$ DNE (jumping up and down)

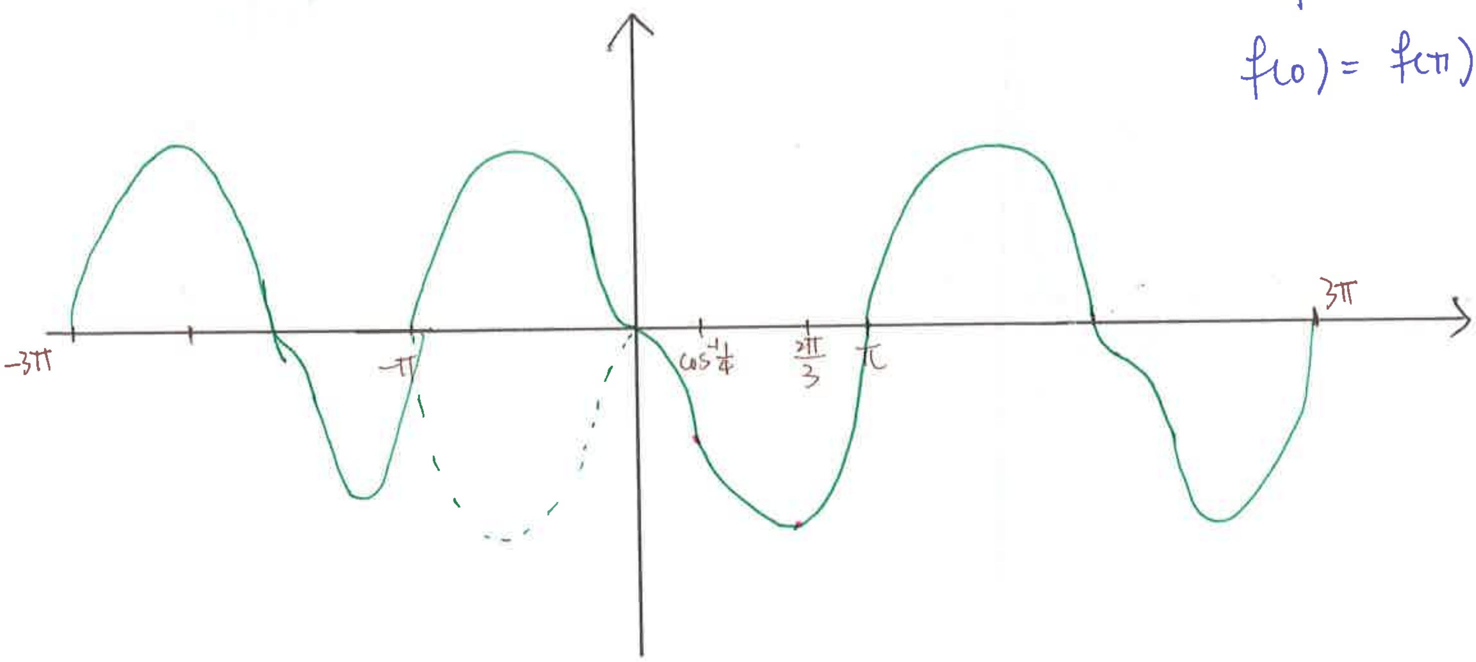


$f(\cos^{-1}\frac{1}{4}) \approx -1.4$

$f(2\pi/3) = -\frac{3}{2}\sqrt{3} \approx -2.6$.

$f = 0 \Rightarrow x = 0, \pi$

$f(0) = f(\pi) = 0$



eg// Economy - Finance Related.

The cost of building a small office is $\$\ 1 + 0.1(n-1)$ - million for n_{TH} floor

and 5 million fixed cost.

If the rent for each floor is 2M/year, how many floors will yield greatest return in investment ?

return = income/cost.

$$\text{Cost.} \quad C(n) = 5 + 1 + 1.1 + \cdots + [1 + 0.1(n-1)]$$

$$= 5 + \frac{1 + 1 + 0.1(n-1)}{2} n = \frac{0.1n^2 + 1.9n + 10}{2}$$

$$I(n) = 2n$$

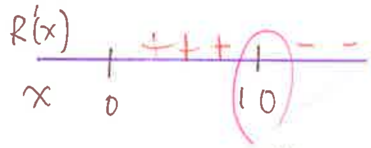
$$\therefore \text{ Return } \quad R(n) = \frac{I(n)}{C(n)} = \frac{4n}{0.1n^2 + 1.9n + 10} \quad : \mathbb{N} \to \mathbb{R}$$

maximize

can't differentiate $R(n)$ $\therefore$ consider $R(x) = \dfrac{4x}{0.1x^2 + 1.9x + 10}$

$$R'(x) = \frac{40 - 0.4x^2}{(0.1x^2 + 1.9x + 10)^2} = 0 \quad \text{when} \quad x = \pm 10$$

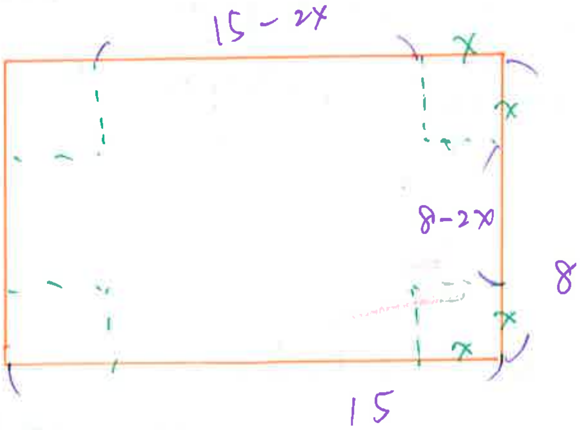
$\rightarrow 10$

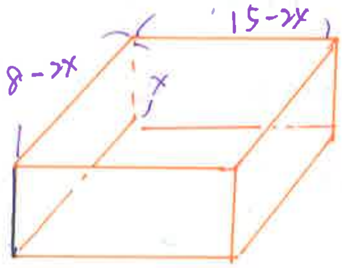


$\therefore$ Build 10 floors for maximum return.

eg. (Exercise 15)

Form a box by cutting 4 identical square corners w/ side x from a $15 \times 8\ \text{cm}^2$ paper and fold. Find x so that volume is maximized





$$0 \le x \le 4$$

$$V(x) = x(15-2x)(8-2x) = 4x^3 - 46x^2 + 120x$$

$$V'(x) = 12x^2 - 92x + 120 = 4(3x-5)(x-6)$$

$$x = \frac{5}{3}, \quad x \neq 6$$

$$V''(x) = 24x - 92 \quad \text{and} \quad V''\left(\frac{5}{3}\right) < 0$$

$$V(0) = V(4) = 0 \qquad V\left(\frac{5}{3}\right) = \frac{5}{3}\left(\frac{35}{3}\right)\left(\frac{14}{3}\right)$$

$$= \frac{2450}{27}$$ ← max volume

$A'(x) = 24 - 16x$

$A'(x) = 0$ at $\frac{3}{2}$

$\therefore$ suspect list : $\{0, \frac{3}{2}, 3\}$

f' : + + + + | - - - - - - -

x : 0 $\quad$ $3/2$ $\quad$ 3

Local max

$A(0) = 0$, $A(3) = 0$

$A(\frac{3}{2}) = 18$

abs. max

$\therefore$ area is maximized at $x = \frac{3}{2}$, with value 18

Hunt/Search for abs. max/min using techniques developed previously

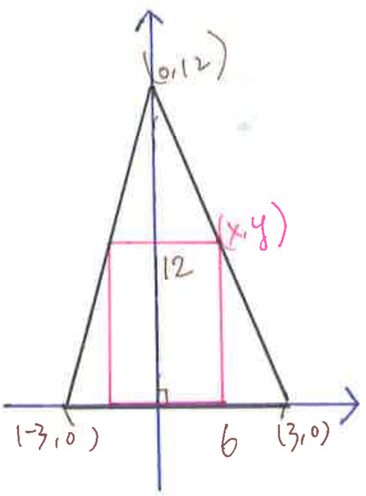
* Max/Min Problems (Optimization)

· Model a real world quatity as a function of some variable and [look for optimum (usually max or min) value] of that quantity and when the quantity is achieved.

the hunt for max/min values.

Geometric Related Problems.

An Isoceles △ has base 6 and height 12. Find the maximum possible area for a rectangle inscribed in it with one side resting on the base. What is its dimensions?



· Draw (/Place) the figure representing the quantities mentioned in the problem on xy plane as conviniently as possible

$A(x, y) = 2xy$

But (x, y) lies on the line $y = -4x + 12$

$\Rightarrow A(x) = 2x(-4x+12)$

$= 24x - 48x^2$

$: [0,3] \to \mathbb{R}$

write down the quantity to be optimized w/ a function of a single variable using provided info.

Recognize the domain of independent variable.

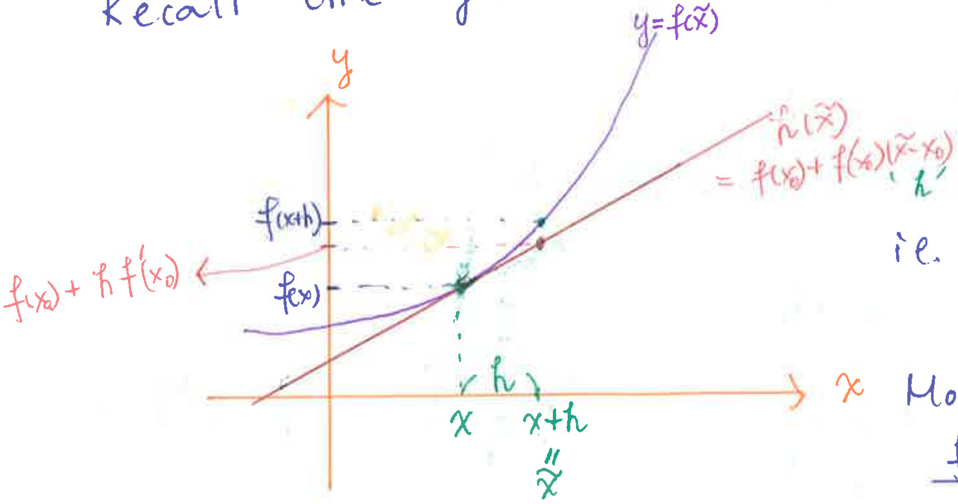
* Numerical Approximations.

Numerical Analysis $\Big\{$ What is numerical value for $\sqrt{102}$? – Differentials

What is the root for $x - \cos x$? – Newton's Method.

Differentials:

Recall the geometric insight of derivatives:



For h small,

$$f(x+h) - f(x) \approx f'(x)h$$

i.e. $f(x+h) - f(x) - f'(x)h \to 0$ as $h \to 0$

Moreover,

$$\frac{f(x+h)-f(x)-f'(x)\cdot h}{h} = \frac{f(x+h)-f(x)}{h} - f'(x)$$

$\to 0$ as $h \to 0$ as well

We say that, h approximates f to the first order at x if

$$f(x) = h(x)$$
$$f'(x) = h'(x)$$

Here $h(t) = f(x) + f'(x)(t-x)$ approximates f to the 1st order

In general, a function h approximates f at x to the n^{th} order if

$$f^{(j)}(x) = h^{(j)}(x) \quad \forall\ 0 \le j \le n.$$

Defn (Differential)

$h \neq 0$, $\Delta f = f(x+h) - f(x)$ (increment of f) is approximated by $df = f'(x)h$, the differential of f at x.

As observed,

$$\delta f - df \to 0 \quad \text{as } h \to 0$$

and $$\frac{\delta f - df}{h} \to 0 \quad \text{as } h \to 0.$$

eg// $f(x) = x^2$

$$\delta f = (x+h)^2 - x^2 = 2xh + h^2$$

$$df = f'(x)h = 2xh$$

$$\delta f - df = h^2 \xrightarrow{h\to 0} 0 \quad ; \quad \frac{\delta f - df}{h} = h \xrightarrow{h\to 0} 0$$

Numerical Applications:

eg// Approximate (a) $\sqrt{104}$ (b) $\cos 40^\circ$.

(a) Take $x = 100$, $f = \sqrt{x}$; $f' = \frac{1}{2\sqrt{x}}$, $h = 4$

$$\sqrt{104} = f(100+4)$$

$x = 100$: $$f(100+4) \approx f(100) + df = \sqrt{100} + \frac{1}{2\sqrt{100}} \cdot 4 = 10 + \frac{1}{5} = 10.2.$$

$$\delta f = f(100+4) - f(100) \approx df(100)$$

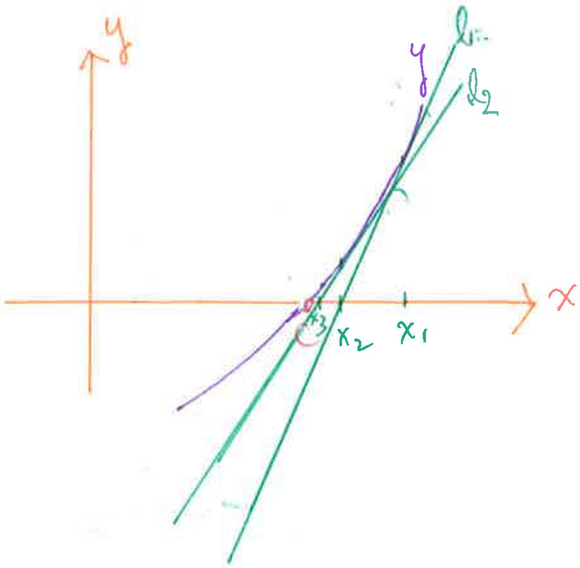
(b) $$\cos 40^\circ = \cos(45^\circ - 5^\circ) = \cos\left(\frac{\pi}{4} - \frac{\pi}{36}\right)$$

$$f = \cos x, \quad x = \frac{\pi}{4}, \quad h = -\frac{\pi}{36}, \quad f' = -\sin x$$

$$\cos 40^\circ = f\left(\frac{\pi}{4} - \frac{\pi}{36}\right) \approx f\left(\frac{\pi}{4}\right) - f'\left(\frac{\pi}{4}\right)\left(-\frac{\pi}{36}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\pi}{36}$$ //

Approximate value of $f(x')$ when there is a x near x' where $f(x)$ is easy to compute

* Newton - Raphson Approximations



Approximate c with an initial guess x_1

l_1: $y - f(x_1) = f'(x_1)(x - x_1)$

$y = 0 \Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

l_2: $y - f(x_2) = f'(x_2)(x - x_2)$

$y = 0 \Rightarrow x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$\vdots$

Get a sequence $\{x_n\}$ with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (*)$$

if $x_n \to x$ as $n \to \infty$

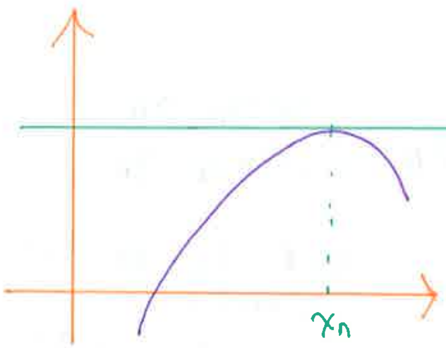
Hopefully, $f(x_n) = f'(x_n)(x_n - x_{n+1})$

$n \to \infty$ $f(x) = 0$.

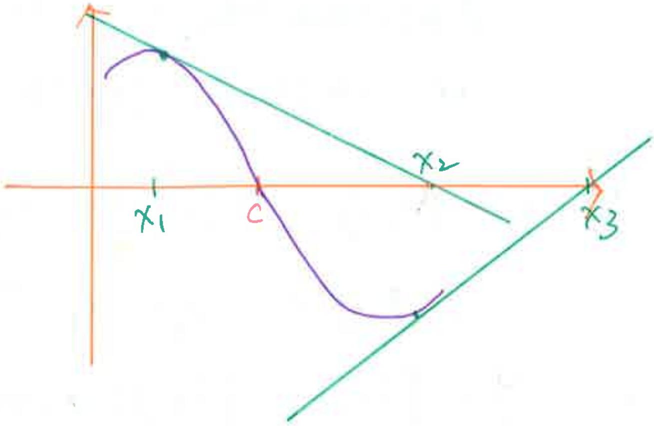
$\therefore$ $x = c$ (under suitable conditions)

There are cases when the approximation fails.

- $f'(x_n) = 0$ for some $n \Rightarrow (*)$ fails, as tangent line never crosses x axis



- $\{x_n\}$ might not converge.



Valid approximation happens when
- f behaves nicely
- x_1 is well chosen.

eg.// Approximate $\sqrt{3}$.

$\sqrt{3}$ is root for function $f(x) = x^2 - 3$.

Pick $x_1 = 2$. $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3}{2x_n}$

$\Rightarrow \quad x_2 \approx 1.75$

$x_3 \approx 1.73$

$\vdots$

eg.// Approximate the root for $f(x) = \cos x - x$

w/ $x_1 = 1$.

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}$$

$$x_2 = 1 - \frac{\cos 1 - 1}{-\sin 1 - 1} \approx 0.75$$

$$x_3 = 0.75 - \frac{\cos 0.75 - 0.75}{-\sin 0.75 - 1} \approx 0.73$$

$$x_4 = 0.73 - \frac{\cos 0.73 - 0.73}{-\sin 0.73 - 1}$$

$$\approx 0.739$$

$\vdots$