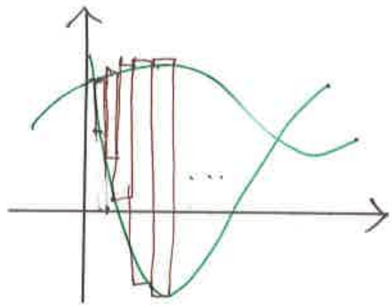
VII. Applications of Integration

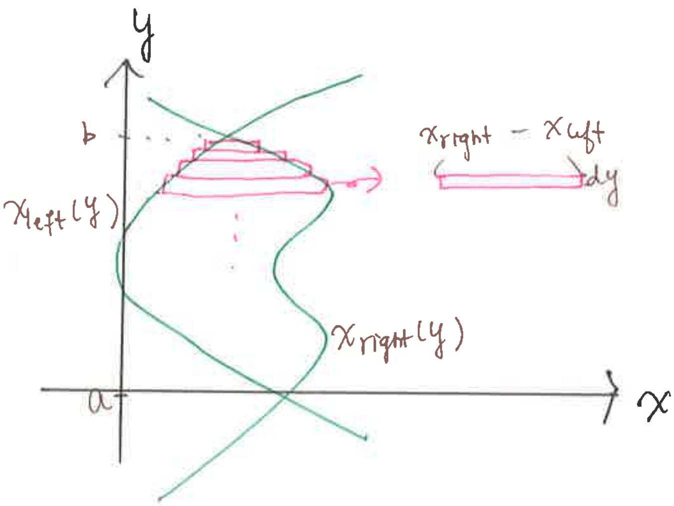
Think of $\int_a^b f(x)\,dx$ as infinite version

of $\sum_{i=0}^{n-1} \underbrace{f(x_i)\,\Delta x_i}_{\text{small part of the entire quantity}}$ $\quad x_0 = a, \quad x_n = b$

We've learned area between curves $y_1(x) = f(x)$ and $y_2(x) = g(x)$, when region is cut into vertical rectangular strips.



When curves are given as functions of y.
$[x_1(y) = f(y) \; ; \; x_2(y) = g(y)]$
cut the region into horizontal strips



$$\text{Area} = \int_a^b (x_{right} - x_{left})\,dy.$$

eg 4 compute area bounded by

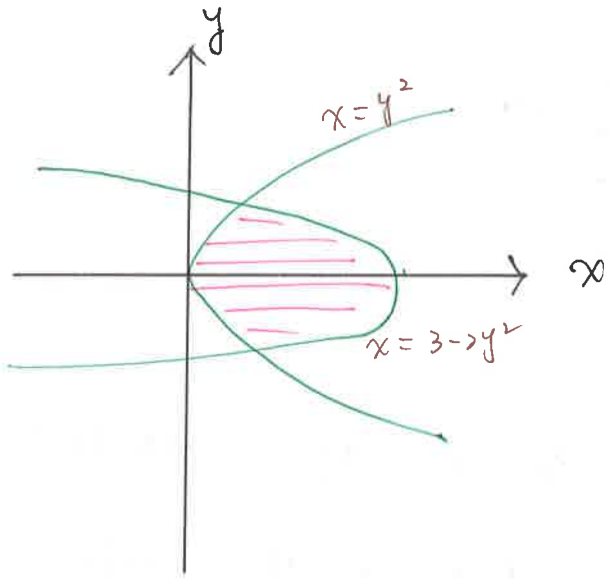
$$x = y^2 \quad \text{and} \quad x = 3 - 2y^2$$

$$y^2 = 3 - 2y^2 \Rightarrow y = \pm 1$$

$y = 0$, $\quad y^2 = 0 < 3 - 2y^2 \big|_{y=0} = 3$

$\therefore \quad x_{left} = y^2 \qquad x_{right} = 3 - 2y^2$

$$A = \int_{-1}^{1} \left[(3 - 2y^2) - y^2 \right] dy = 4$$



eg. compute area bounded by $x = y^2$ and $x = y+2$

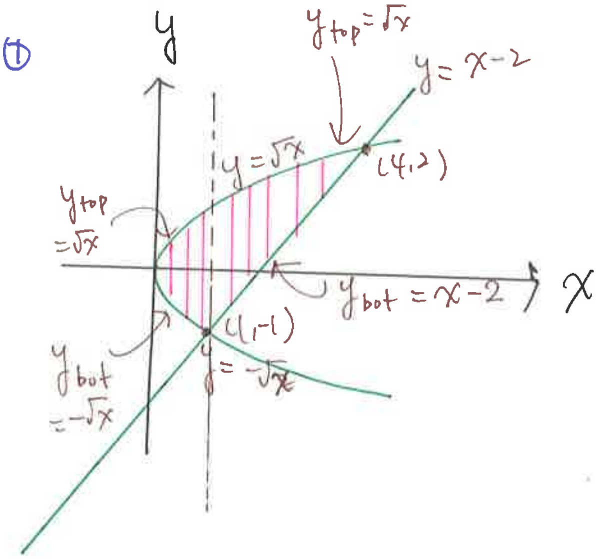
by ① integrating wrt. x

② " " y

$$y^2 = y+2 \Rightarrow y = -1, 2 \Rightarrow x = 1, 4$$

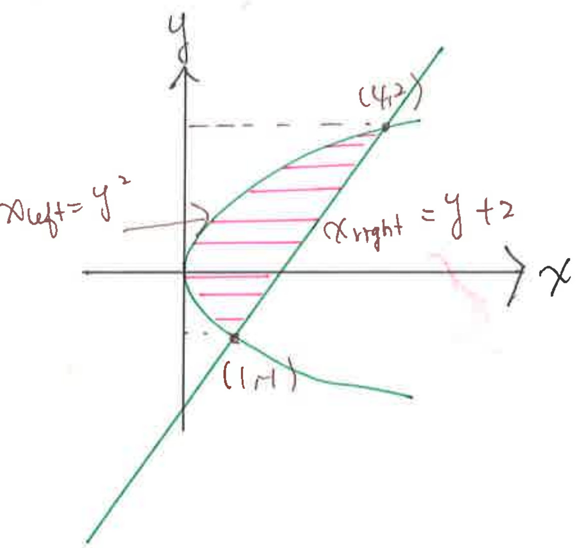
points of intersection $(1,-1)$, $(4,2)$.

①



$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 [\sqrt{x} - (x-2)]\,dx$$
$$= \frac{9}{2}$$

②



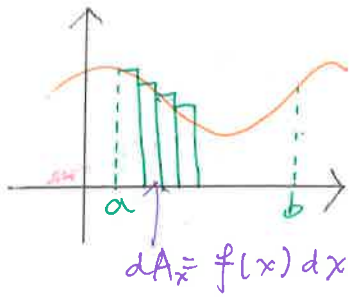
$$A = \int_{-1}^{2} [(y+2) - y^2]\,dy = \frac{9}{2}$$

* Volume by Cross Sections

Revisiting the idea of integration, we summarize, integration is a computation of a quantity by adding up small parts

$$\underset{\text{area}}{A} = \underset{\text{sum of}}{\int_a^b} \underbrace{dA_x}_{\text{small area}}$$

lower and upper limits



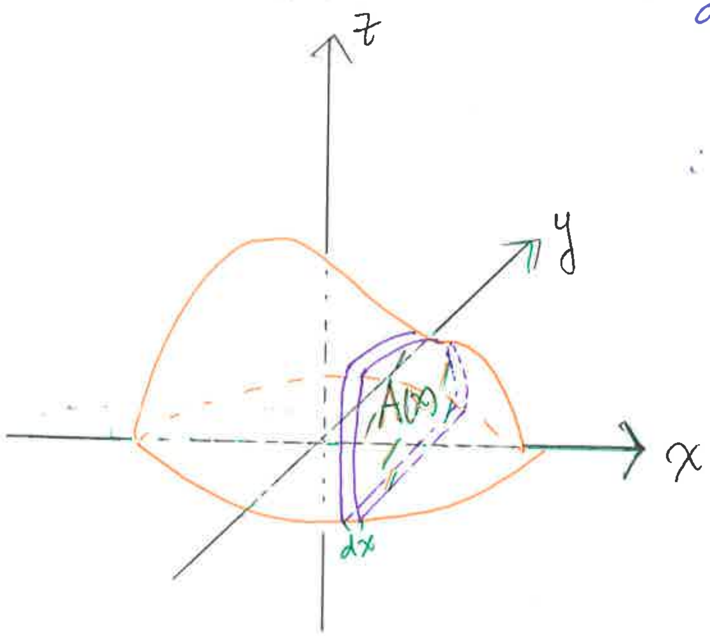
Apply this principle to more general quantities:

we study volume of a solid in $\mathbb{R}^3$.

$$V = \int_a^b dV_x$$

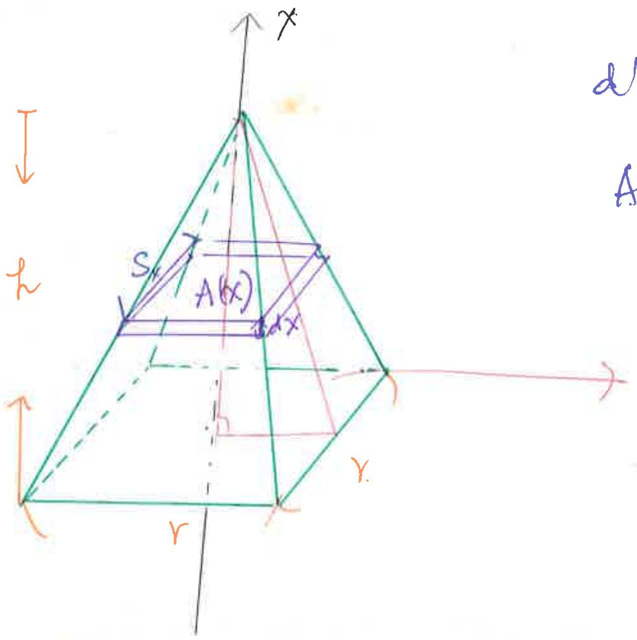
describe small volume as a function of x

$$dV_x = A(x)\,dx$$



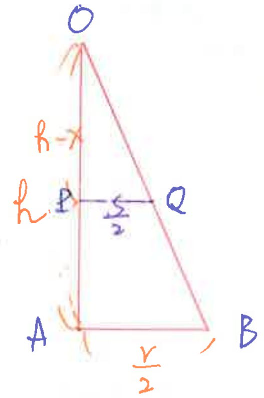
$\therefore$ need to determine $A(x)$ as a function of x in order to perform the integration.

eg 11 Find the volume of the pyramid of height h, whose base is a square of length r. ⑥



$dV_x = A(x)\,dx$

$A(x) = S_x^2$



$\triangle OPQ \sim \triangle OAB$

$\therefore \quad \frac{\overline{OP}}{\overline{OA}} = \frac{\overline{PQ}}{\overline{AB}}$

$\frac{h-x}{h} = \frac{\frac{s}{2}}{\frac{r}{2}}$

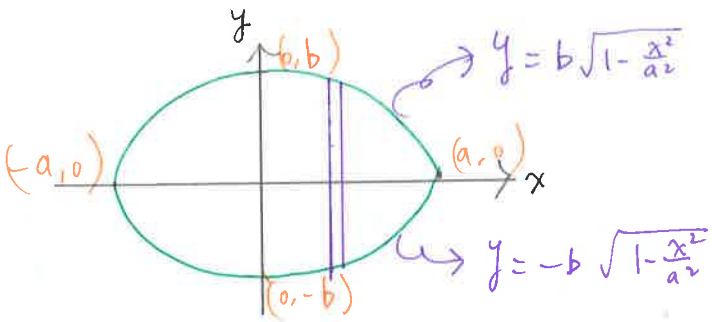
$\Rightarrow \quad S_x = \frac{r}{h}(h-x)$

$$\therefore \quad V = \int_0^h A(x)\,dx$$

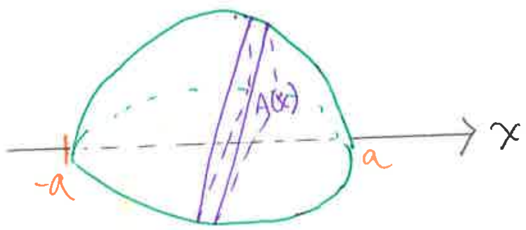
$$= \frac{r^2}{h^2}\int_0^h (h-x)^2\,dx$$

$$= \frac{r^2}{h^2}\,\frac{-(h-x)^3}{3}\Big|_0^h = \frac{1}{3}r^2h$$

Recall, the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



eg. Find the volume of the solid with base the ellipse as above and cross sections are isosceles triangles whose height is $\frac{1}{2}$ the base.

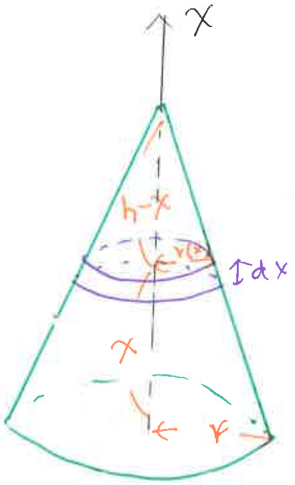


$$dV_x = A(x)\,dx$$

$$A(x) = \frac{1}{2}\left(2b\sqrt{1-\frac{x^2}{a^2}}\right)\left(b\sqrt{1-\frac{x^2}{a^2}}\right)$$
$$= \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore \quad V = \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)\,dx$$
$$= \frac{4}{3}ab^2$$

eg: volume of a circular cone with base a disc of radius r and height h



$$\frac{r(x)}{r} = \frac{h-x}{h} = 1 - \frac{x}{h}$$

$$\Rightarrow \quad r(x) = r - \frac{r}{h} x$$

$$A(x) = \pi r(x)^2 = \pi r^2 \left(1 - \frac{x}{h}\right)^2$$

$$= \pi r^2 \left(1 - \frac{2x}{h} + \frac{x^2}{h^2}\right)$$

$$V = \pi r^2 \int_0^h \left(1 - \frac{2x}{h} + \frac{x^2}{h^2}\right) dh$$

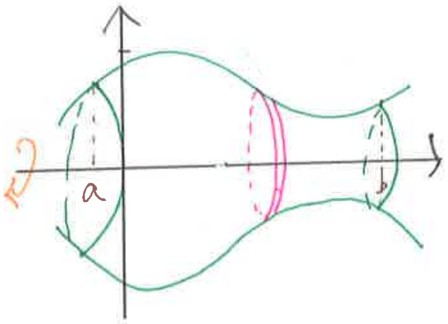
$$= \frac{1}{3} \pi r^2 h$$

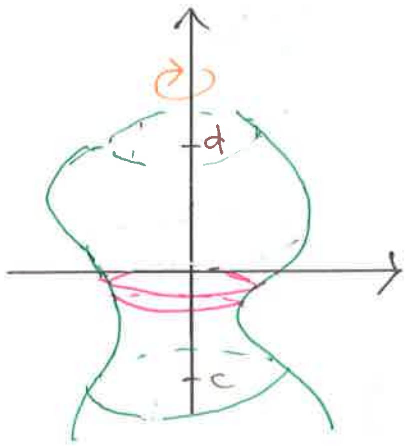
//

Solid of Revolutions - Disk Method

consider the solids formed by rotating curves

$\begin{cases} y = f(x) & \text{around } x\text{-axis} \\ x = g(y) & \text{" } y\text{-axis.} \end{cases}$





$$V = \int_{\sim}^{\sim} dV_x \text{ (or } y)$$

2/5 ends

$$dV_x = \pi [f(x)]^2 dx$$

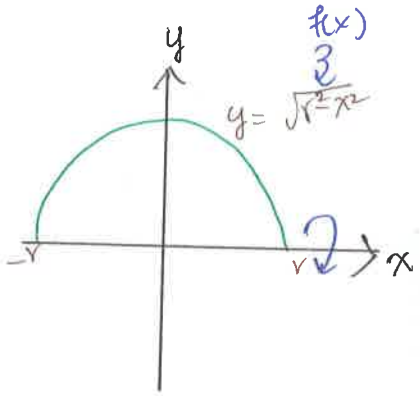
$$V = \int_a^b \pi [f(x)]^2 dx$$

$$dV_y = \pi [g(y)]^2 dy$$

$$V = \int_c^d \pi [g(y)]^2 dy$$

eg// Volume of a ball of radius r

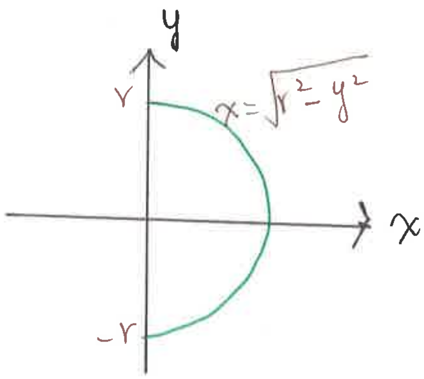
around x-axis:



$$V = \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = 2\pi r^3 - \frac{2r^3}{3} = \frac{4\pi r^3}{3}$$

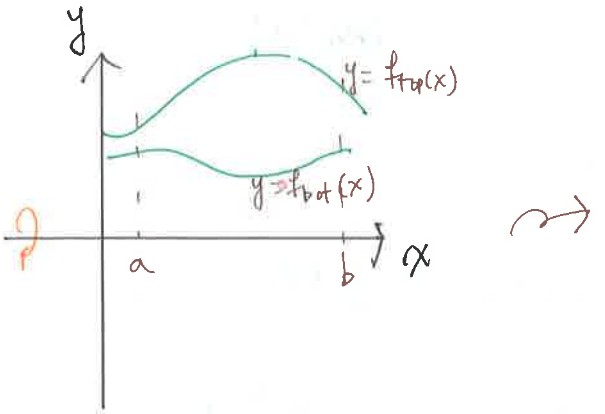
around y-axis

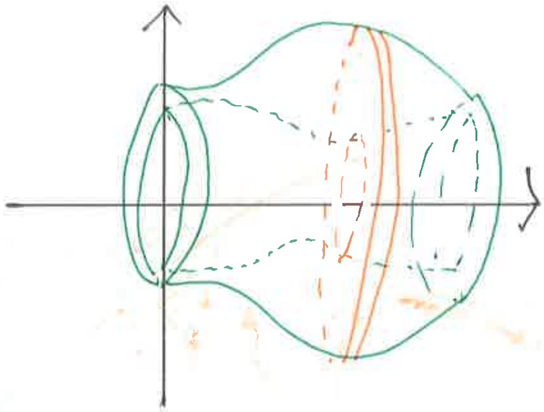


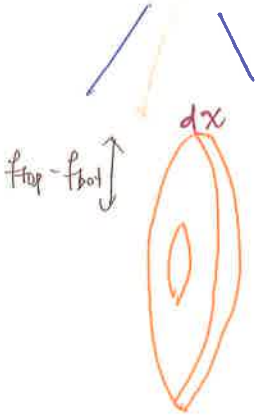
$$V = \int_{-r}^{r} \pi (r^2 - y^2)\, dy = \frac{4\pi r^3}{3}$$

eg//

* Solids of Revolutions of Regions between Curves. (Washer Method)







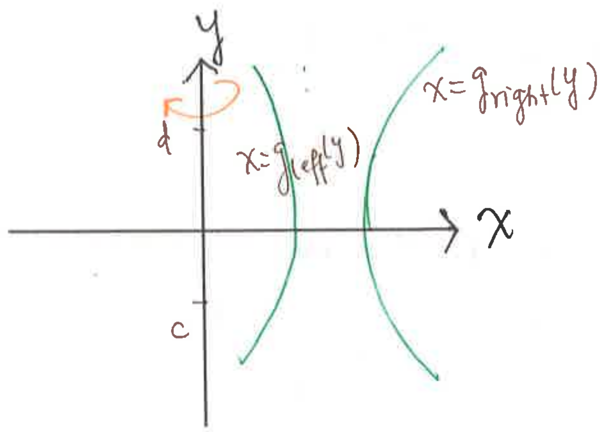
a "washer"

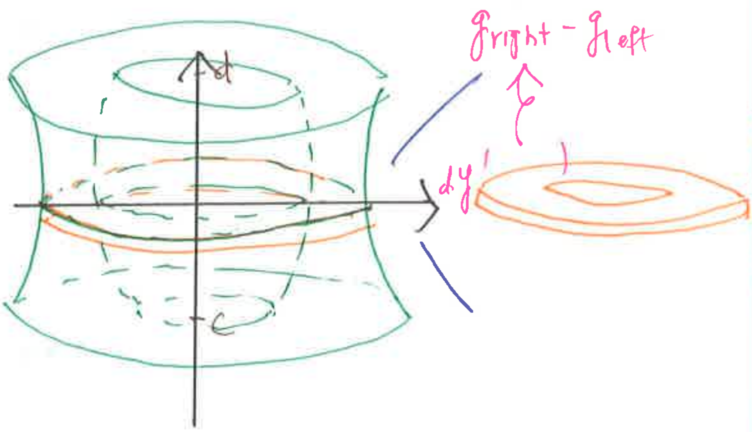
dV_x
$= \pi (f_{top} - f_{bot})^2 dx$

$$V = \int_a^b \pi (f_{top} - f_{bot})^2 dx$$

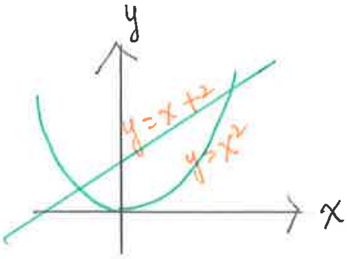
Similarly,
For curves as functions of y, revolved around y axis,

$$V = \int_c^d \pi (g_{right} - g_{left})^2 dy$$





eg,, $y = x^2$; $y = x+2$, around x-axis ⑫



$x^2 = x+2$

$\Rightarrow x = -1, 2 \rightarrow y = 1, 4.$

$$V = \int_{-1}^{2} \pi\left[(x+2)^2 - x^4\right] dx$$

$$= \pi \int_{-1}^{2} (x^2+4x+4-x^4)\, dx$$

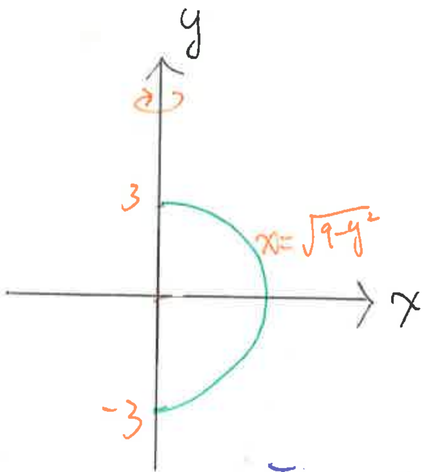
$$= \pi \left(\frac{x^3}{3} + 2x^2 + 4x - \frac{4}{15}x^5 \right) \Big|_{-1}^{2}$$

$$= \pi\left[\left(\frac{8}{3} + 8 + 8 - \frac{128}{5}\right) - \left(-\frac{1}{3} + 2 - 4 + \frac{4}{5}\right)\right]$$

$$= \pi\left(18 - \frac{327}{15}\right)$$ //

eg,, $x = \sqrt{9-y^2}$, $x=0$ around y axis

$\sqrt{9-y^2} = 0$ at $y = \pm 3$



$$V = \int_{-3}^{3} \pi (9-y^2)\, dy$$

$$= \pi \left(9y - \frac{y^3}{3}\right) \Big|_{-3}^{3}$$

$$= \pi\left[(27-9) - (-27+9)\right]$$

$$= 36\pi$$

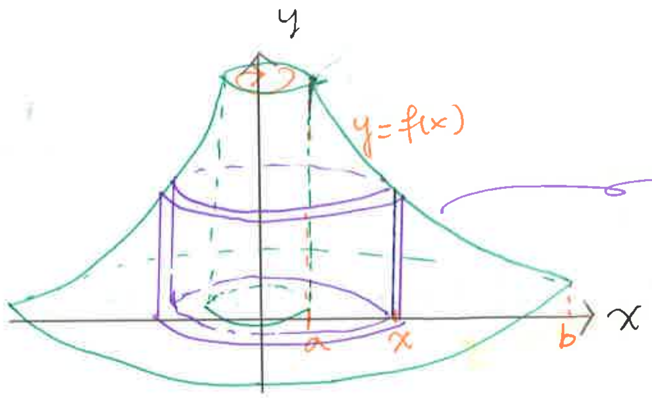
In , we form a ball of radius 3

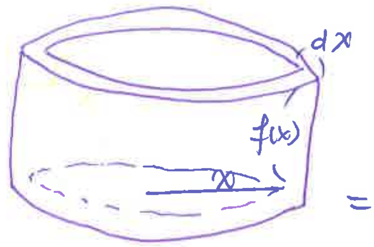
$\therefore$ volume $= \frac{4}{3}\pi \cdot 3^3 = 36\pi$ //

* Volume By Shell Method

Consider different ways of cutting the solid.

Revolving $y=f(x)$ around y axis



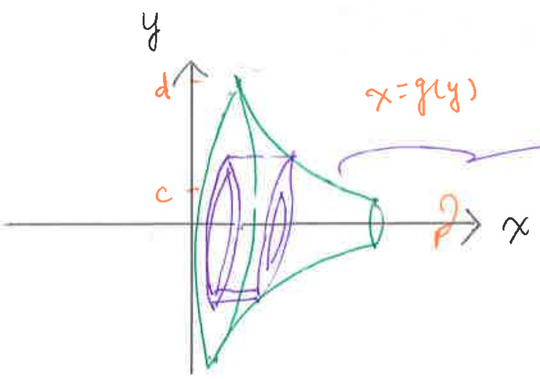


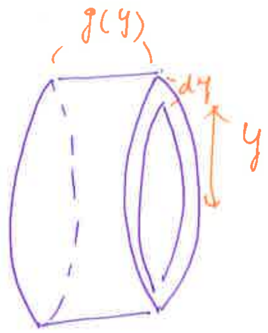
dV_x

$= 2\pi x f(x) dx$

$$V = \int_a^b dV_x = \int_a^b 2\pi x f(x) dx$$

Similarly, we may revolve $x=g(y)$ around x axis

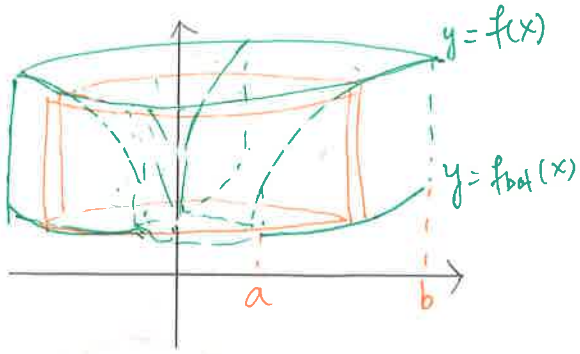


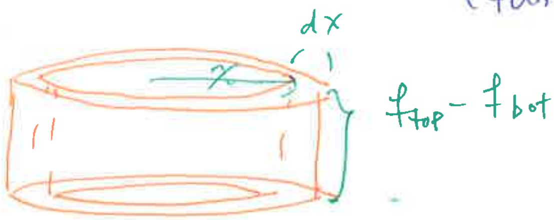


$$V = \int_c^d 2\pi y g(y)\, dy$$

Revolving area bounded by curves ...

around y-axis
(func. of x)





$$V = \int_a^b 2\pi x \left[f_{top}(x) - f_{bot}(x) \right] dx$$

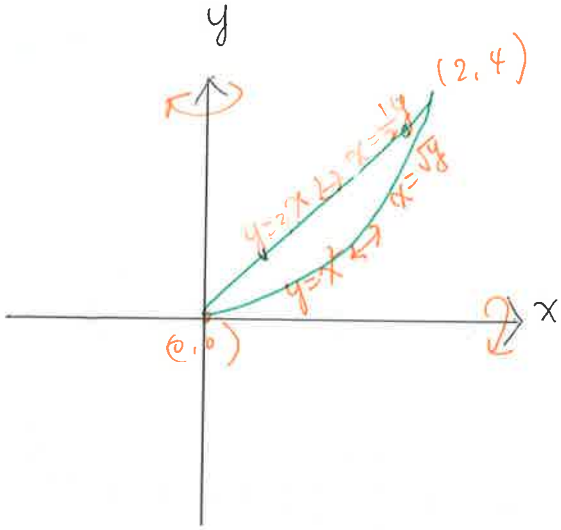
around x-axis
(func. of y)

$$V = \int_c^d 2\pi y \left[g_{right}(y) - g_{left}(y) \right] dy$$

eg. Find the volume of the solid generated by revolving the region between

$y = x^2$ and $y = 2x$

ⓐ around y-axis ⓑ around x-axis



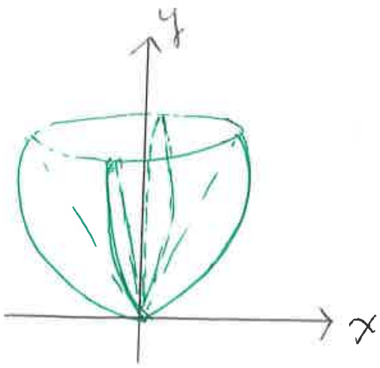
$y = x^2$

$x^2 = 2x$

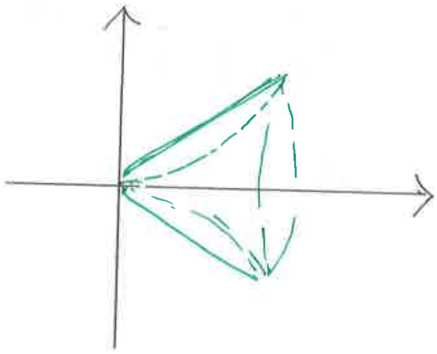
$\rightarrow x = 0, 2$

$\rightarrow y = 0, 4$

ⓐ



ⓑ



$$V = \int_0^2 2\pi(2x - x^2)x \, dx$$

$$= \frac{8}{3}\pi$$

$$V = \int_0^4 2\pi\left(\sqrt{y} - \frac{1}{2}y\right)y \, dy$$

$$= \frac{64}{15}\pi$$

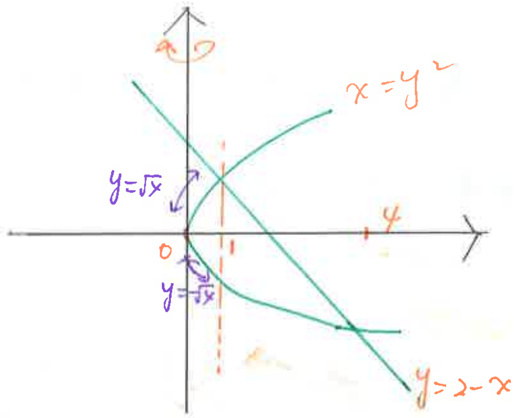
eg11 $x = y^2$, $x = 2-y$ around y axis

$y^2 = 2-y$

$\rightarrow y^2 + y - 2 = 0$

$y = -2, 1$

$\rightarrow x = 4, 1$



$$V = 2\pi\int_0^1 x[\sqrt{x} - (-\sqrt{x})]\,dx + 2\pi\int_1^4 [2-x-(-\sqrt{x})]\,x\,dx$$

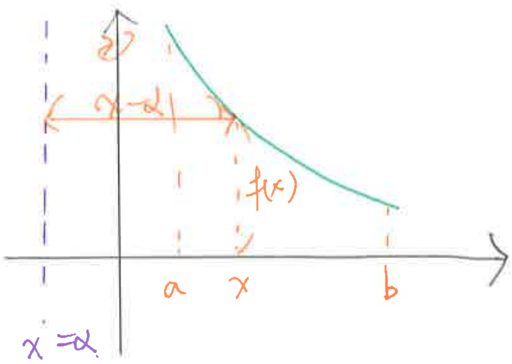
$$= 2\pi\int_0^1 2x^{\frac{3}{2}}\,dx + 2\pi\int_1^4 (2x - x^2 + x^{\frac{3}{2}})\,dx$$

$$= \frac{8\pi}{5} + \left(16 - \frac{64}{3} + \frac{72}{5}\right) - \left(1 - \frac{1}{3} + \frac{2}{5}\right)$$

$$= \frac{8\pi}{5} + 8$$ //

17

Revolving around shifted axis. $x=\alpha$

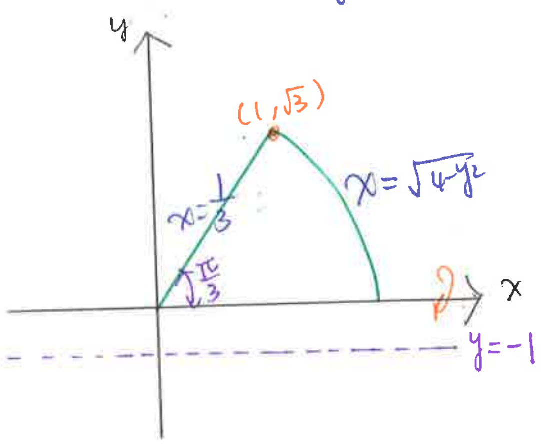


$$V=\int_a^b 2\pi f(x)\cdot(x-\alpha)\,dx$$

Similarly, for solid formed by revolution around $y=\beta$

$$V=\int_c^d 2\pi g(y)\,(y-\beta)\,dy$$

eg11 Axis of revolution $y=-1$



$$x=\frac{1}{3}y\ ;\quad x=\sqrt{4-y^2}$$

$$V=2\pi\int_0^{\sqrt{3}}\left(\sqrt{4-y^2}-\frac{1}{3}y\right)(y+1)\,dy$$

$$=2\pi\int_0^{\sqrt{3}}\sqrt{4-y^2}\cdot y\,dy-\frac{2\pi}{3}\int_0^{\sqrt{3}}y^2\,dy$$

I II

$$+\ 2\pi\int_0^{\sqrt{3}}\left(\sqrt{4-y^2}-\frac{1}{3}y\right)dy$$

III

$$V=\frac{2\pi}{3}\left(\frac{7}{3}-\sqrt{3}+\frac{2\pi}{3}\right)$$

$u=4-y^2$

$$\mathrm{I}=\int_4^1\sqrt{u}\cdot y\,\frac{dy}{-2y}=\frac{1}{3}\sqrt{u^3}\Big|_1^4=\frac{7}{3}\ ;\quad \mathrm{II}=\sqrt{3}$$

$$\mathrm{III}=\pi(2^2)\cdot\frac{\frac{\pi}{3}}{2\pi}=\frac{2\pi}{3}$$