HW#2 Solutions

§ 2.3

(20) $\lim_{x\to -2} \frac{(x^2-x-6)^2}{x+2} = \lim_{x\to -2} \frac{(x+2)^{\cancel{2}}(x-3)^2}{\cancel{x+2}} = 0$ //

(25) $\lim_{h\to 0} \frac{1-1/h^2}{1-1/h} = \lim_{h\to 0} \frac{h^2-1}{h^2-h} = \lim_{h\to 0} \frac{(h+1)\cancel{(h-1)}}{h\cancel{(h-1)}}$ DNE

"=" $\frac{1}{0}$

(26) $\lim_{h\to 0} \frac{1-1/h^2}{1+1/h^2} = \lim_{h\to 0} \frac{h^2-1}{h^2+1} = -1$ //

(27) $\lim_{h\to 0} \frac{1-1/h}{1+1/h} = \lim_{h\to 0} \frac{h-1}{h+1} = -1$ //

(28) $\lim_{h\to 0} \frac{1+1/h}{1+1/h^2} = \lim_{h\to 0} \frac{h^2+h}{h^2+1} = 0$ //

(38) $\lim_{x\to -4} \left(\frac{2x}{x+4} - \frac{8}{x+4}\right) = \lim_{x\to -4} \left(\frac{2x-8}{x+4}\right)$

$= \lim_{x\to -4} \frac{2(x-4)}{x+4}$ "=" $\frac{-16}{0}$ DNE

(42) $f(x) = x^3$

(a) $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{x^3-3^3}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}}$

$= 27$ //

(b) $\lim_{x\to 3} \frac{f(x)-f(2)}{x-3}$ DNE // since $f(3)-f(2) \neq 0$.

(c) $\lim_{x\to 3} \frac{f(x)-f(3)}{x-2} = \frac{f(3)-f(3)}{1} = 0$

(d) $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$, $f(x)-f(1) = x^3-1^3$

$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 3$ //

(52) ⓐ if $f(x) \geq g(x)$, $|f(x) - g(x)| = f(x) - g(x)$

and $\frac{1}{2}\{[f(x) + g(x)] + |f(x) - g(x)|\} = f(x)$

if $f(x) < g(x)$, $|f(x) - g(x)| = g(x) - f(x)$

and " " $= g(x)$

which are precisely definition of $\max\{f(x), g(x)\}$

i.e. $\max\{f(x), g(x)\} = \begin{cases} f(x) : f(x) \geq g(x) \\ g(x) ; f(x) < g(x) \end{cases}$

ⓑ $\min\{f(x), g(x)\} = \begin{cases} f(x) : f(x) < g(x) \\ g(x) : f(x) \geq g(x) \end{cases}$

$\therefore$ $\frac{1}{2}\{[f(x) + g(x)] - |f(x) - g(x)|\}$ will do.

(check it!)

(53)

$\lim h(x) = \lim_{x \to c} \frac{1}{2}\{[f(x) + g(x)] - |f(x) - g(x)|\}$

$|\ |$ is continuous and commutes w/ limit $\rightarrow$ $\stackrel{?}{=} \frac{1}{2}\{\lim_{x \to c} f(x) + \lim_{x \to c} g(x) - |\lim_{x \to c} f(x) - \lim_{x \to c} g(x)|\}$ $\nearrow 0$

$= \frac{1}{2}(L + L) = L$

Similarly $\lim_{x \to c} H(x) = \lim_{x \to c} \frac{1}{2}\{f(x) + g(x) + |f(x) - g(x)|\}$

$= \frac{1}{2}\{L + L + |\lim_{x \to c} f(x) - \lim_{x \to c} g(x)|\} = L$ //

$y = 3x,\quad x \to 0 \Leftrightarrow y \to 0$

(6) $\displaystyle \lim_{x\to 0} \frac{\sin(3x)}{5x} = \frac{3}{5} \lim_{x \to 0} \frac{\sin(3x)}{3x} = \frac{3}{5} \lim_{y\to 0} \frac{\sin y}{y} = \frac{3}{5}$ //

(8) let $y = x^2$, $y \to 0 \Leftrightarrow x \to 0$

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y \to 0} \frac{\sin y}{y} = 1 \text{ //}$$

(14) $\displaystyle \lim_{x\to 0} \frac{4x}{\cot(3x)} = \lim_{x\to 0} \frac{4x}{\cos(3x)} \cdot \sin(3x)$

$$= \frac{0}{1} \cdot 0 = 0 \text{ //}$$

(22) $\displaystyle \lim_{x\to 0} \frac{1-\cos(4x)}{9x^2} = \lim_{x\to 0} \left[\frac{1-\cos(4x)}{9x^2} \cdot \frac{1+\cos(4x)}{1+\cos(4x)} \right]$

$$= \lim_{x\to 0} \left[\frac{1-\cos^2(4x)}{9x^2} \cdot \frac{1}{1+\cos(4x)} \right]$$

$$= \frac{16}{9} \lim_{x\to 0} \frac{\sin^2(4x)}{(4x)^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos(4x)} = \frac{1}{2}$$

$$= \frac{8}{9} \Big[\underbrace{\lim_{x\to 0} \frac{\sin(4x)}{4x}}_{=1} \Big]^2 = \frac{8}{9} \text{ //}$$

(32) $\displaystyle \sin\left(x + \frac{\pi}{3}\right) = \sin\left(x - \frac{\pi}{6} + \frac{\pi}{2}\right) = \sin\left(x - \frac{\pi}{6}\right) \overset{0}{\cos\frac{\pi}{2}} + \cos\left(x - \frac{\pi}{6}\right) \overset{1}{\sin\frac{\pi}{2}}$

$$\lim_{x\to \frac{\pi}{6}} \frac{\sin(x+\frac{\pi}{3}) - 1}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}} \frac{\cos(x - \frac{\pi}{6}) - 1}{x - \frac{\pi}{6}} = \lim_{y\to 0} \frac{\cos y - 1}{y} = 0 \text{ //}$$

$y = x - \frac{\pi}{6}$

$x \to \frac{\pi}{6} \Leftrightarrow y \to 0$

(36) $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b} \lim_{x\to 0}\left[\frac{\sin ax}{ax} \cdot \frac{bx}{\sin bx}\right]$

$$= \frac{a}{b} \frac{\lim_{x\to a} \frac{\sin(ax)}{ax} = 1}{\lim_{x\to 0} \frac{\sin(bx)}{bx} = 1} = \frac{a}{b}$$ //

(43)

$0 \le \left|x \sin\frac{1}{x}\right| = |x| \left|\sin\frac{1}{x}\right| \le |x|$ since $0 \le \left|\sin\frac{1}{x}\right| \le 1$

$0 \xrightarrow{x\to 0} 0$, $\quad |x| \xrightarrow{x\to 0} 0$

By pinching thm., $\lim_{x\to 0} \left|x \sin\frac{1}{x}\right| = 0$

and by corollary ~~in~~ class. $\lim_{x\to 0} \left(x \sin\frac{1}{x}\right) = 0$. //

(44) Again

$0 \le \left|(x-\pi) \cos^2\left(\frac{1}{x-\pi}\right)\right| = |x-\pi| \left(\cos^2\left(\frac{1}{x-\pi}\right)\right) \le \underbrace{|x-\pi|}$

$0 \le \cos^2\left(\frac{1}{x-\pi}\right) \le 1$

$0 \xrightarrow{x\to\pi} 0$, $\quad |x-\pi| \xrightarrow{x\to\pi} 0$

Pinching Thm $\Rightarrow \lim_{x\to\pi} \left|(x-\pi) \cos^2\left(\frac{1}{x-\pi}\right)\right| = 0$

Corollarly $\Rightarrow \lim_{x\to\pi} (x-\pi) \cos^2\left(\frac{1}{x-\pi}\right) = 0.$