Hw 3. § 2-4 # 8, 12, 22, 36, 42

§ 2-6 # 4, 12, 25, 28.

§ 2-4

8. $h(x) = \begin{cases} x^2+5 & , x<2 \\ x^3 & , x \geq 2 \end{cases}$, $x=2$

$\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} (x^3) = 8$ $\quad$ $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} (x^2+5) = 9$

$\lim_{x\to 2^+} h(x) \neq \lim_{x\to 2^-} h(x)$, thus, $x=2$ is jump discontinuity.

12. $f(x) = \begin{cases} 1-x & , x<1 \\ 1 & , x=1 \\ x^2-1 & , x>1 \end{cases}$, $x=1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2-1) = 1^2-1 = 0$ $\quad$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$

$\lim_{x\to 1} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 0$, but $f(1) = 1$

, thus, $x=1$ is removable discontinuity.

22.

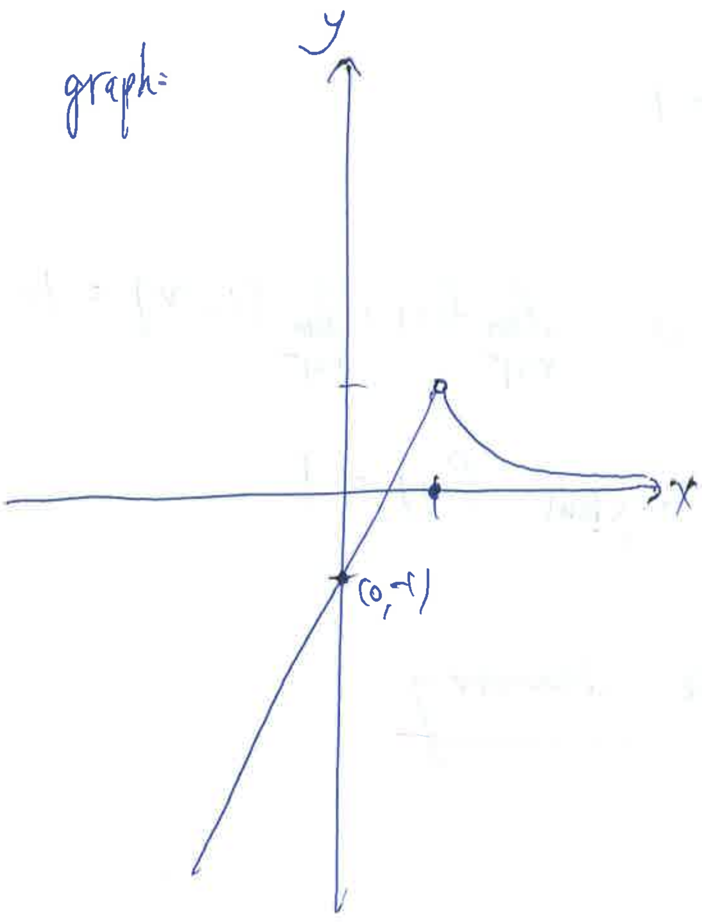
$$g(x) = \begin{cases} 2x-1, & x<1 \\ 0, & x=1 \\ \frac{1}{x^2}, & x>1 \end{cases}$$

$\lim_{x\to1^+} g(x) = \lim_{x\to1^+} \left(\frac{1}{x^2}\right) = \frac{1}{1^2} = 1$ $\lim_{x\to1^-} g(x) = \lim_{x\to1^-} (2x-1) = 1$

$\lim_{x\to1} g(x) = \lim_{x\to1^+} g(x) = \lim_{x\to1^-} g(x) = 1$, but $g(1)=0$.

thus, $x=1$ is removable discontinuity.

graph:



§ 2-4

3

36.

$$f(x)=\begin{cases} A^2x^2, & x\le 2 \\ (1-A)x, & x>2 \end{cases}$$

find $A=$ ____ satisfies f is continuous at 2.

since $\lim_{x\to 2} f(x) = f(2)$ holds.

$f(2) = A^2 \cdot 2^2 = 4A^2$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (1-A)x = 2(1-A) = 2-2A$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} A^2x^2 = 4A^2 \Rightarrow 2-2A = 4A^2$

$\Rightarrow 2A^2+A-1=0$

$\Rightarrow (2A-1)(A+1)=0$

$\Rightarrow A=\frac{1}{2}$ or $A=-1$

$A=\frac{1}{2} \Rightarrow \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 1$ and $f(2)=1$

$\Rightarrow \lim_{x\to 2} f(x) = 1 = f(2) \Rightarrow f$ is continuous at 2

$A=-1 \Rightarrow \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4$ and $f(2)=4$

$\Rightarrow \lim_{x\to 2} f(x) = 4 = f(2) \Rightarrow f$ is continuous at 2

thus, $A=\frac{1}{2}$ or $A=-1$

42.

$$f(x) = \frac{\sqrt{x+4}-3}{\sqrt{x-5}}$$

$f(5)$ is undefined.

$$f(x) = \frac{\sqrt{x+4}-3}{\sqrt{x-5}} = \frac{(\sqrt{x+4}-3)(\sqrt{x+4}+3)}{\sqrt{x-5}(\sqrt{x+4}+3)} = \frac{\overset{\sqrt{x-5}}{\cancel{x-5}}}{\cancel{\sqrt{x-5}}(\sqrt{x+4}+3)}$$

$$f(x) = \frac{\sqrt{x-5}}{\sqrt{x+4}+3}$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{\sqrt{x-5}}{\sqrt{x+4}+3} = \frac{\sqrt{5-5}}{\sqrt{5+4}+3} = \frac{0}{6} = 0$$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{\sqrt{x-5}}{\sqrt{x+4}+3} = \frac{\sqrt{5-5}}{\sqrt{5+4}+3} = \frac{0}{6} = 0.$$

$\Rightarrow \lim_{x\to 5} f(x) = 0$, but $f(5)$ is undefined.

Let define: $f(5) = 0$, then $\lim_{x\to 5} f(x) = 0 = f(5)$

$\Rightarrow$ $f(x)$ is continuous at 5.

4. Intermediate value theorem

$2\tan x - x = 1$, $[0, \frac{\pi}{4}]$

<pf>

$f(x) = 2\tan x - x - 1$, $0 \le x \le \frac{\pi}{4}$, is continuous.

$$\begin{cases} f(0) = 2\tan 0 - 0 - 1 = -1 < 0 \\ f(\frac{\pi}{4}) = 2\tan\frac{\pi}{4} - \frac{\pi}{4} - 1 = 2\times 1 - \frac{\pi}{4} - 1 = 1 - \frac{\pi}{4} \fallingdotseq 0.22 > 0. \end{cases}$$

$(\tan\frac{\pi}{4} = 1)$

since $f(0) < 0 < f(\frac{\pi}{4})$, $\exists\, c \in (0, \frac{\pi}{4})$, such that $f(c) = 0$.

$f(c) = 2\tan c - c - 1 = 0 \Rightarrow 2\tan c - c = 1$

12. Show that there exists $c > 0$ such that $c^3 = 2$.

<pf> Let $f(x) = x^3 - 2$ is continuous on $\mathbb{R}$.

$f(0) = -2$, $f(1) = 1^3 - 2 = -1$, $f(2) = 8 - 2 = 6$

So, $f(x) = x^3 - 2$ is continuous, on $[1, 2]$

$f(1) = -1 < 0$, $f(2) = 6 > 0$

since $f(1) < 0 < f(2)$, $\exists\, c \in (1, 2) \Rightarrow c > 0$, such that $f(c) = 0$

$\Rightarrow f(c) = c^3 - 2 = 0 \Rightarrow c^3 = 2$

25. (Fixed point property)

if f is continuous on $[0,1]$ and $0 \le f(x) \le 1$

then there exists at least one point c in $[0,1]$, such that $f(c) = c$.

<pf>

Let $g(x) = x - f(x)$

clearly, $g(x) = x - f(x)$ is continuous on $[0,1]$

(continuous $\pm$ continuous = continuous)

$g(0) = 0 - f(0) = -f(0) \le 0$, since $0 \le f(0) \le 1 \Rightarrow -1 \le -f(0) \le 0$

$g(1) = 1 - f(1) \ge 0$, since $0 \le f(1) \le 1 \Rightarrow -1 \le -f(1) \le 0$

$\Rightarrow 0 \le 1 - f(1) \le 1$

by intermediate-value theorem, $\exists\, c \in (0,1)$ such that $g(c) = 0$.

$\Rightarrow g(c) = c - f(c) = 0 \Rightarrow f(c) = c$.

28. Show that every real number a, there exists a number c such that $c^3 = a$.

(cube root)

<pf> Let $f(x) = x^3 - a$ is continuous on $\mathbb{R}$.

we always choose two real number $x_1 < x_2$ such that $f(x_1) < 0$, $f(x_2) > 0$

so, $f(x)$ is continuous on $[x_1, x_2]$ and $f(x_1) \le 0 \le f(x_2)$

by intermediate-value theorem, $\exists\, c \in (x_1, x_2)$ such that $f(c) = 0$

$\Rightarrow f(c) = c^3 - a = 0 \Rightarrow c^3 = a$