HW4.

§3-1 ※ 14, 20, 22, 24, 30, 35, 41, 44, 46.

§3-2 ※ 6, 9, 12, 20, 24, 28, 30, 32, 36, 40, 44, 50, 54, 58, 66.

§3-3 ※ 8, 18, 26, 30, 40, 50, 53, 60, 64

§3-1

14. Find $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

$f(x) = 5 - x^4$, $c = -1$

$f(c+h) = f(-1+h) = 5 - (-1+h)^4 = 5 - (h^4 - 4h^3 + 6h^2 - 4h + 1) = -h^4 + 4h^3 - 6h^2 + 4h + 4$

$f(c) = f(-1) = 5 - (-1)^4 = 4$

$\frac{f(-1+h) - f(-1)}{h} = \frac{-h^4 + 4h^3 - 6h^2 + 4h + 4 - 4}{h} = -h^3 + 4h^2 - 6h + 4$

$\lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} (-h^3 + 4h^2 - 6h + 4) = 4$

20. $f(x) = 5 - x^3$, $c = 2$,

tangent line: $y = f'(c)(x - c) + f(c)$

$f(2) = 5 - 2^3 = 5 - 8 = -3$, $f(2+h) = 5 - (2+h)^3 = 5 - h^3 - 12h - 6h^2 - 8 = -h^3 - 6h^2 - 12h - 3$

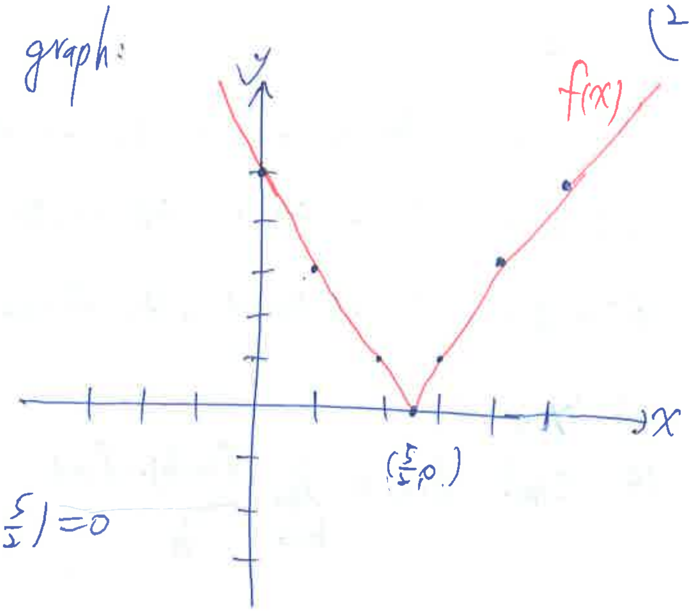
$\frac{f(2+h) - f(2)}{h} = \frac{5 - (2+h)^3 - (-3)}{h} = -h^2 - 6h - 12$,

$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} (-h^2 - 6h - 12) = -12$

tangent line: $y = -12(x - 2) + (-3) \Rightarrow y = -12x + 21 \Rightarrow 12x + y - 21 = 0$

24. $f(x) = |2x-5|$

graph:



x	$\cdots$	0	1	2	$\frac{5}{2}$	3	4	5	$\cdots$
$f(x)$	$\cdots$	5	3	1	0	1	3	5	$\cdots$

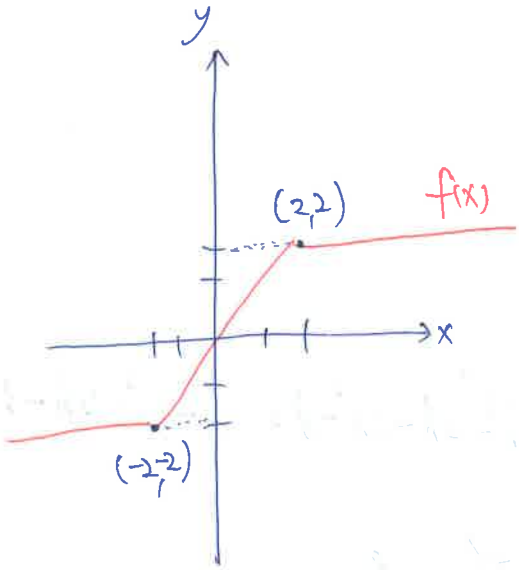
by graph of $f(x)$,

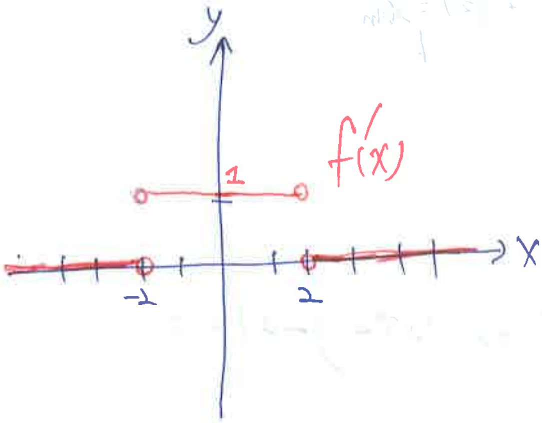
$f(x)$ is not differentiable at $c = \frac{5}{2}$, $f(\frac{5}{2}) = 0$

$$f'(\tfrac{5}{2}) = \lim_{h \to 0} \frac{f(\frac{5}{2}+h) - f(\frac{5}{2})}{h} \qquad f(\tfrac{5}{2}+h) = |2(\tfrac{5}{2}+h) - 5| = |2h|$$

$$= \lim_{h \to 0} \frac{|2h| - 0}{h} = \lim_{h \to 0} \frac{|2h|}{h} \text{ does not exist. } \left(\text{since } \lim_{h \to 0^+} \frac{|2h|}{h} = 2, \ \lim_{h \to 0^-} \frac{|2h|}{h} = -2 \right)$$

35.





by the graph of $f(x)$,

$$f(x) = \begin{cases} 2, & x \geq 2 \\ x, & -2 \leq x \leq 2 \\ -2, & x \leq -2 \end{cases}$$

$$f'(x) = \begin{cases} 0, & x > 2 \\ 1, & -2 < x < 2 \\ 0, & x < -2 \end{cases}$$

and $f(x)$ is not differentiable at $x = 2, -2$.

41.

$$f(x) = \begin{cases} x^3, & x \le 1 \\ Ax+B, & x > 1 \end{cases}$$ is differentiable at $x=1$, Find A, B

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$f(1) = 1^3 = 1$, $f(1+h) = A(1+h)+B$, as $h>0$

$f(1+h) = (1+h)^3 = h^3+3h^2+3h+1$, as $h<0$

$$\lim_{h \to 0^+} \frac{A(1+h)+B-1}{h} = \lim_{h \to 0^+} \frac{A+B+Ah-1}{h}$$

$$\lim_{h \to 0^-} \frac{h^3+3h^2+3h+1-1}{h} = \lim_{h \to 0} (h^2+3h+3) = 3$$

since $f'(1)$ exists, then, we have

$$\lim_{h \to 0^+} \frac{A+B+Ah-1}{h} = 3$$

Let $A+B-1=0$, we have $\lim_{h \to 0^+} \frac{Ah}{h} = \lim_{h \to 0^+} A = A = 3 \Rightarrow B = -2$

$A=3$, $B=-2$

44. $$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

as $c \neq 0$, $f(c) = 1$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

as $h>0$ such that $c+h \neq 0 \Rightarrow f(c+h) = 1$

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0$$

as $h<0$ such that $c+h \neq 0 \Rightarrow f(c+h) = 1$

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{1-1}{h} = 0$$

$f'(c) = 0$, as $c \neq 0$.

as $c=0$, $f(0)=0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{\substack{h \to 0 \\ (h \neq 0)}} \frac{f(h)}{h} = \lim_{h \to 0} \frac{1}{h}$$ does not exist.

§3-2

※6. $F(x)=\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2-x$

$F'(x)=\frac{1}{4}\cdot4\cdot x^3-\frac{1}{3}\cdot3\cdot x^2+\frac{1}{2}\cdot2\cdot x-1$

$=\underline{x^3-x^2+x-1}$

9. $G(x)=(x^2-1)(x-3)$, Let $f(x)=x^2-1$ $\quad g(x)=x-3$

$G'(x)=f'(x)\cdot g(x)+f(x)\cdot g'(x)$

$=2x\cdot(x-3)+(x^2-1)\cdot1$

$=2x^2-6x+x^2-1$

$=\underline{3x^2-6x-1}$

20. $G(x)=(1+\frac{1}{x})(1+\frac{1}{x^2})$, Let $f(x)=1+\frac{1}{x}$ $\quad g(x)=1+\frac{1}{x^2}$

$f'(x)=\frac{0-1}{x^2}=\frac{-1}{x^2}$ $\qquad g'(x)=\frac{0-2x}{x^4}=\frac{-2}{x^3}$

$G'(x)=f'(x)\cdot g(x)+f(x)\cdot g'(x)$

$=\frac{-1}{x^2}\cdot(1+\frac{1}{x^2})+(1+\frac{1}{x})\cdot\frac{-2}{x^3}$

$=\frac{-1}{x^2}-\frac{1}{x^4}-\frac{2}{x^3}-\frac{2}{x^4}$

$=\underline{\frac{-1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}}$

※24、 $f(x)=\dfrac{2x^2+x+1}{x^2+2x+1}$ find $f'(0)$ and $f'(1)$

$$f'(x)=\frac{(x^2+2x+1)\cdot(2x^2+x+1)'-(2x^2+x+1)\cdot(x^2+2x+1)'}{(x^2+2x+1)^2}$$

$$=\frac{(x^2+2x+1)\cdot(4x+1)-(2x^2+x+1)\cdot(2x+2)}{(x^2+2x+1)^2}$$

$$f'(0)=\frac{1\times1-1\times2}{1^2}=\underline{-1} \qquad f'(1)=\frac{4\times5-4\times4}{4^2}=\frac{4}{16}=\underline{\frac{1}{4}}$$

28、 $h(0)=3$, $h'(0)=2$, find $f'(0)=?$

$$f(x)=3x^2\cdot h(x)-5x$$

$$f'(x)=6x\cdot h(x)+3x^2\cdot h'(x)-5$$

$$f'(0)=6\cdot0\cdot\underbrace{h(0)}_{=3}+3\cdot0^2\cdot\underbrace{h'(0)}_{=2}-5=\underline{-5}$$

30、 $h(0)=3$, $h'(0)=2$, find $f'(0)=?$

$$f(x)=h(x)+\frac{x}{h(x)}$$

$$f'(x)=h'(x)+\frac{h(x)\cdot1-x\cdot h'(x)}{[h(x)]^2}=h'(x)+\frac{h(x)-x\cdot h'(x)}{[h(x)]^2}$$

$$f'(0)=h'(0)+\frac{h(0)-0\cdot h'(0)}{(h(0))^2}=2+\frac{3-0\cdot2}{3^2}=2+\frac{3}{9}=\underline{\frac{7}{3}}$$

32.

$f(x) = (x^3-2x+1)\cdot(4x-5)$, $c=2$,

$f'(x) = (3x^2-2)\cdot(4x-5) + 4\cdot(x^3-2x+1)$

$f'(2) = 10\cdot 3 + 4\cdot 5 = 30+20 = 50.$, $f(2) = 5\times 3 = 15$

tangent line:

$y = f'(2)(x-2) + f(2)$

$y = 50(x-2)+15 \Rightarrow y = 50x - 100 + 15$, $y = 50x - 85$

36. tangent line is horizontal $\Rightarrow f'(c) = 0.$

$f(x) = x^2 - \frac{16}{x}$ $\qquad f'(x) = 2x - \frac{0-16}{x^2} = 2x + \frac{16}{x^2}$

Let $f'(x) = 0 \Rightarrow 2x + \frac{16}{x^2} = 0$, $x \neq 0$,

$2x = -\frac{16}{x^2}$

$x = \frac{-8}{x^2}$

$x^3 = -8$

$x = -2$, $(-2, f(-2)) = (-2, 12)$

#40.

$f(x) = 3x^4 - 4x^3 - 2$

$f'(x) = 12x^3 - 12x^2$

(a)

Let $f'(x)=0 \Rightarrow 12x^3 - 12x^2 = 0$

$12(x^3 - x^2) = 0$

$12x^2(x-1) = 0$

$x = 0$ or $x = 1$

$x<0$	$0<x<1$	$1<x$
$f'(x)<0$	$f'(x)<0$	$f'(x)>0$

(b)

$f'(x) > 0 \Rightarrow 12x^2(x-1) > 0 \quad \because x^2 \geq 0$

$x - 1 > 0$

$x > 1$, that is, $(1, \infty)$

(c) $f'(x) < 0 \Rightarrow 12x^2(x-1) < 0$

$x-1 < 0$

$x < 1$, that is, $(-\infty, 1)$, but $f'(0) = 0$

so, $(-\infty, 0) \cup (0, 1)$

44. $f(x) = x^3 - 3x$ is perpendicular to the line $5y - 3x = 8$

$f'(x) = 3x^2 - 3$ and the slope of the line $5y - 3x = 8$ is $\frac{3}{5}$

since the tangent line perpendicular to the line, so the slope of tangent line is $\frac{-5}{3}$ ($\because \frac{3}{5} \times \frac{-5}{3} = -1$)

Let $f'(x) = 3x^2 - 3 = \frac{-5}{3}$

$9x^2 - 9 = -5$

$9x^2 = 4$

$x^2 = \frac{4}{9}$

$x = \frac{2}{3}$ or $\frac{-2}{3}$, that is, $\left(\frac{2}{3}, \frac{-46}{27}\right)$

$\left(\frac{-2}{3}, \frac{46}{27}\right)$

50.

$$f(x)=\begin{cases} Ax^2+B, & x<-1 \\ Bx^5+Ax+4, & x\geq -1 \end{cases}$$ and $f'(x)$ is continuous everywhere.

$\Rightarrow$ $f(x)$ is continuous at $x=-1$ and $f'(x)$ is continuous at $x=-1$

$\Rightarrow \lim_{x\to -1} f(x) = f(-1)$ $\quad f(-1)=B\cdot(-1)^5+A(-1)+4=-B-A+4$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (Bx^5+Ax+4) = B\cdot(-1)^5+A\cdot(-1)+4 = -B-A+4$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (Ax^2+B) = A\cdot(-1)^2+B = A+B$

since $\lim_{x\to -1} f(x)$ exists, then $-B-A+4=A+B$

$2A+2B=4$

$A+B=2$ —— ①

$\Rightarrow \lim_{x\to -1} f'(x) = f'(-1)$

$$f'(x)=\begin{cases} 2Ax & x<-1 \\ 5Bx^4+A, & x\geq -1 \end{cases}$$ $\quad f'(-1)=5B\cdot(-1)^4+A=A+5B.$

$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+} (5Bx^4+A) = 5B+A$ $\quad \lim_{x\to -1^-} f'(x) = \lim_{x\to -1^-} (2Ax) = -2A$

since $\lim_{x\to -1} f'(x)$ exists, then $5B+A=-2A \Rightarrow 3A+5B=0$ —— ②

by ① and ②, we have $A=5$

$B=-3$

54.

$f(x) = Ax^3 + Bx^2 + Cx + D$ is tangent to the line $y = 3x - 3$ at $(1, 0)$

$\Rightarrow f'(1) = 3$ and $f'(x) = 3Ax^2 + 2Bx + C$ and $f(1) = 0$

$\Rightarrow 3A + 2B + C = 3$ and $A + B + C + D = 0$

$f(x) = Ax^3 + Bx^2 + Cx + D$ is tangent to the line $y = 18x - 27$ at $(2, 9)$

$\Rightarrow f'(2) = 18$ and $f(2) = 9$

$\Rightarrow 12A + 4B + C = 18$ and $8A + 4B + 2C + D = 9$

$$\begin{cases} 3A + 2B + C = 3 \\ A + B + C + D = 0 \\ 12A + 4B + C = 18 \\ 8A + 4B + 2C + D = 9 \end{cases} \Rightarrow \begin{matrix} A = 3 & C = 6 \\ B = -6 & D = -3 \end{matrix}$$

66.

<pf> $[f(x)\cdot g(x)\cdot h(x)]' = [(f(x)\cdot g(x))\cdot h(x)]'$ (use the product rule)

$= (f(x)\cdot g(x))'\cdot h(x) + h'(x)\cdot(f(x)\cdot g(x))$

use the product rule

$= (f'(x)g(x) + f(x)\cdot g'(x))\cdot h(x) + h'(x)\cdot f(x)\cdot g(x)$

$= f'(x)\cdot g(x)h(x) + f(x)\cdot g'(x)\cdot h(x) + h'(x)\cdot f(x)\cdot g(x)$

Q.E.D.

§3-3

※8.

$$y=\left(\frac{x}{1+x}\right)\cdot\left(\frac{2-x}{3}\right)$$

$$\frac{dy}{dx}=\left(\frac{x}{1+x}\right)'\cdot\frac{2-x}{3}+\left(\frac{2-x}{3}\right)'\cdot\frac{x}{1+x}$$

$$=\frac{(1+x)-x}{(1+x)^2}\cdot\frac{2-x}{3}+\frac{-\cancel{3}^{1}-(2-x)\cdot 0}{\cancel{9}\,3}\cdot\frac{x}{1+x}$$

$$=\frac{2-x}{3(1+x)^2}+\frac{-x}{3(x+1)}$$

$$=\frac{2-x}{3(1+x)^2}+\frac{-x(x+1)}{3(x+1)^2}$$

$$=\frac{2-x-x^2-x}{3(1+x)^2}=\underline{\frac{-x^2-2x+2}{3(1+x)^2}}$$

18.

$$\frac{d}{du}\left(\frac{u^2}{u^3+1}\right)$$

$$=\frac{(u^3+1)\cdot 2u-u^2\cdot 3u^2}{(u^3+1)^2}$$

$$=\frac{2u^4+2u-3u^4}{(u^3+1)^2}$$

$$=\underline{\frac{-u^4+2u}{(u^3+1)^2}}$$

※ 26. $y = \dfrac{(x^2+1)(x^2-2)}{x^2+2}$, $x=2$

$$\frac{dy}{dx} = \frac{(x^2+2)\cdot[(x^2+1)(x^2-2)]' - (x^2+1)(x^2-2)\cdot(x^2+1)'}{(x^2+2)^2}$$

$$= \frac{(x^2+2)[2x(x^2-2)+2x\cdot(x^2+1)] - (x^2+1)(x^2-2)\cdot 2x}{(x^2+2)^2}$$

$$= \frac{(x^2+2)\cdot(4x^3-2x) - (x^2+1)(x^2-2)\cdot 2x}{(x^2+2)^2}$$

$x=2$ 代入

$$\Rightarrow \frac{dy}{dx} = \frac{6\times 28 - 5\times 2\times 4}{36} = \frac{128}{36} = \frac{32}{9}$$

40. $\dfrac{d^2}{dx^2}\left[(x^2-3x)\cdot\dfrac{d}{dx}(x+x^{-1})\right]$

$$= \frac{d^2}{dx^2}\left[(x^2-3x)\cdot(1-x^{-2})\right]$$

$$= \frac{d}{dx}\left[(2x-3)\cdot(1-x^{-2}) + 2x^{-3}\cdot(x^2-3x)\right]$$

$$= \frac{d}{dx}\left[2x - \cancel{2x^{-1}} - 3 + 3x^{-2} + \cancel{2x^{-1}} - 6x^{-2}\right]$$

$$= 2 + \cancel{2x^{-2}} - 0 + (-6x^{-3}) + (\cancel{-2x^{-2}}) + 12x^{-3}$$

$$= 2 + 6x^{-3} = 2 + \frac{6}{x^3}$$

53.

$$f(x)=\begin{cases} x^2, & x\geq 0. \\ 0, & x<0 \end{cases}$$

(a)

<pf>

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}, \quad f(0)=0$$

$$=\lim_{h\to 0}\frac{f(h)}{h}$$

$$\lim_{h\to 0^+}\frac{f(h)}{h}=\lim_{h\to 0^+}\frac{h^2}{h}=\lim_{h\to 0^+}h=0$$

$$\lim_{h\to 0^-}\frac{f(h)}{h}=\lim_{h\to 0^-}\frac{0}{h}=0$$

then $f'(0)=\lim_{h\to 0}\frac{f(h)}{h}=0$

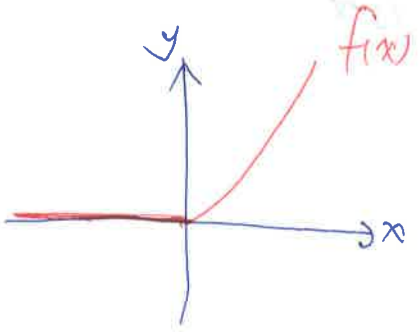
(b)

$$f'(x)=\begin{cases} 2x, & x>0 \\ 0, & x\leq 0 \end{cases}$$

(c) $f''(0)=\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0}\frac{f'(h)}{h}$ does not exist. ($\because 2\neq 0$)

$$\lim_{h\to 0^+}\frac{2h}{h}=\lim_{h\to 0^+}2=2, \quad \lim_{h\to 0^-}\frac{0}{h}=0.$$

(d)



60.

$f(x) = x^4 + 3x^3 - 6x^2 - x$

$f'(x) = 4x^3 + 9x^2 - 12x - 1$

$f''(x) = 12x^2 + 18x - 12$

(a) Let $f''(x) = 0 \Rightarrow 12x^2 + 18x - 12 = 0$

$2x^2 + 3x - 2 = 0$

$(2x-1)(x+2) = 0$

$x = \frac{1}{2}$ or $x = -2$

(b) Let $f''(x) > 0 \Rightarrow (2x-1)\cdot(x+2) > 0$

$x > \frac{1}{2}$ or $x < -2$

(c) Let $f''(x) < 0 \Rightarrow (2x-1)(x+2) < 0$

$-2 < x < \frac{1}{2}$

64. (a) $\frac{d^n}{dx^n}(x^n) = \frac{d^{n-1}}{dx^{n-1}}(n \cdot x^{n-1}) = \frac{d^{n-2}}{dx^{n-2}}(n \cdot (n-1) \cdot x^{n-2})$

$\vdots$

$= n \cdot (n-1) \cdot (n-2) \cdots \cdot 2 \cdot 1 = n!$

(b) $\frac{d^{n+1}}{dx^{n+1}}(x^n) = \frac{d}{dx}\left(\frac{d^n}{dx^n}(x^n)\right) = \frac{d}{dx}(\underset{\substack{\uparrow \\ \text{constant}}}{n!}) = 0$