Hw6.

§3-7: 12, 22, 56

§4-1: 14, 20, 24

§4-2: 6, 16, 24, 34, 42, 56

§3-7

12.

$$x^2-2xy+4y^2=3$$

$$\frac{d(x^2-2xy+4y^2)}{dx}=\frac{d(3)}{dx}$$

$$\frac{dx^2}{dx}-\frac{d(2xy)}{dx}+\frac{d4y^2}{dx}=0$$

$$2x-\left[\frac{d2x}{dx}\cdot y+2x\cdot\frac{dy}{dx}\right]+\frac{d4y^2}{dy}\cdot\frac{dy}{dx}=0$$

$$2x-2y-2x\cdot\frac{dy}{dx}+8y\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{2y-2x}{8y-2x}=\frac{x-y}{x-4y}$$

$$\frac{d\left(2x-2y-2x\cdot\frac{dy}{dx}+8y\cdot\frac{dy}{dx}\right)}{dx}=\frac{d(0)}{dx}$$

$$\frac{d2x}{dx}-\frac{d2y}{dx}-\left[\frac{d2x}{dx}\cdot\frac{dy}{dx}+2x\cdot\frac{d^2y}{dx^2}\right]+\left[\frac{d8y}{dx}\cdot\frac{dy}{dx}+8y\cdot\frac{d^2y}{dx^2}\right]=0$$

$$2-2\cdot\frac{dy}{dx}-2\frac{dy}{dx}-2x\cdot\frac{d^2y}{dx^2}+8\cdot\left(\frac{dy}{dx}\right)^2+8y\cdot\frac{d^2y}{dx^2}=0$$

$$(8y-2x)\cdot\frac{d^2y}{dx^2}=-2+4\cdot\frac{dy}{dx}-8\cdot\left(\frac{dy}{dx}\right)^2$$

$$=-2+4\cdot\frac{x-y}{x-4y}-8\cdot\left(\frac{x-y}{x-4y}\right)^2$$

$$=\frac{-6(x^2-2xy+4y^2)}{(x-4y)^2}=\frac{-18}{(x-4y)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{-18}{(8y-2x)(x-4y)^2}=\frac{-9}{(4y-x)^3}$$

22.

$$9x^2+4y^2=72\quad,\ \text{point } (2,3)$$

$$\frac{d(9x^2+4y^2)}{dx}=\frac{d(72)}{dx}$$

$$\frac{d9x^2}{dx}+\frac{d4y^2}{dx}=0$$

$$18x+\frac{d4y^2}{dy}\cdot\frac{dy}{dx}=0$$

$$18x+8y\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-18x}{8y}=\frac{-9x}{4y}$$

$$\left.\frac{dy}{dx}\right|_{(2,3)}=\frac{-9\times2}{4\times3}=\frac{-3}{2}\ \text{(the slope of tangent line)}$$

$\therefore$ tangent line: $y=\frac{-3}{2}(x-2)+3$

$$\frac{-3}{2}\times m=-1$$

$m=\frac{2}{3}$ (the slope of normal line)

$\therefore$ normal line: $y=\frac{2}{3}(x-2)+3$

※ Show that the sum of the x- and y-intercepts of any line tangent to the graph of $x^{1/2} + y^{1/2} = c^{1/2}$ is constant and equal to c.

<pf> $x^{1/2} + y^{1/2} = c^{1/2}$

$$\frac{d x^{1/2}}{dx} + \frac{d y^{1/2}}{dx} = \frac{d c^{1/2}}{dx}$$

$$\frac{1}{2} \cdot x^{-\frac{1}{2}} + \frac{1}{2} \cdot y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2} \cdot x^{-1/2}}{\frac{1}{2} \cdot y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}} = -\left(\frac{y}{x}\right)^{1/2}$$

the slope of tangent line at (x_0, y_0).

tangent line to the graph at point (x_0, y_0) is $y = -\underbrace{\left(\frac{y_0}{x_0}\right)^{1/2}} \cdot (x - x_0) + y_0$

$x=0 \Rightarrow y = -\left(\frac{y_0}{x_0}\right)^{1/2}(-x_0) + y_0 = x_0^{1/2} y_0^{1/2} + y_0$ (∴ y-intercept)

$y=0 \Rightarrow 0 = -\left(\frac{y_0}{x_0}\right)^{1/2} \cdot (x - x_0) + y_0 \Rightarrow x - x_0 = \frac{y_0}{\left(\frac{y_0}{x_0}\right)^{1/2}} = x_0^{1/2} y_0^{1/2}$

$\Rightarrow x = x_0 + x_0^{1/2} \cdot y_0^{1/2}$ (x-intercept)

$$\therefore x_0 + x_0^{1/2} y_0^{1/2} + y_0 + x_0^{1/2} y_0^{1/2} = x_0 + 2 x_0^{1/2} y_0^{1/2} + y_0$$
$$= \left(x_0^{1/2} + y_0^{1/2}\right)^2$$
$$= \left(c^{1/2}\right)^2$$
$$= c$$

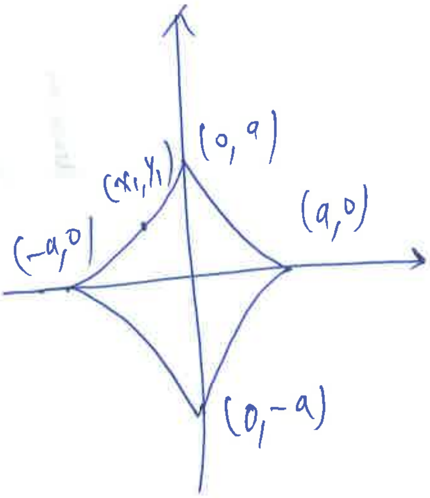
∎

3-7

*56. The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is called an "asteroid." The curve is shown in the figure.

(a) Find the slope of the graph at an arbitrary point (x_1, y_1) which is <u>not</u> a vertex.

(b) At what points of the curve is the slope of the tangent line 0, 1, −1 ?



(a)

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}\cdot y^{-\frac{1}{3}}\cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$m = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}}$$

(b)

$m = 0 = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} \Rightarrow y_1 = 0,\ x_1 = \pm a$ $\quad\therefore (a, 0), (-a, 0)$

$m = 1 = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} \Rightarrow y_1 = -x_1 \Rightarrow x_1 = \pm\frac{1}{4}a\sqrt{2}$

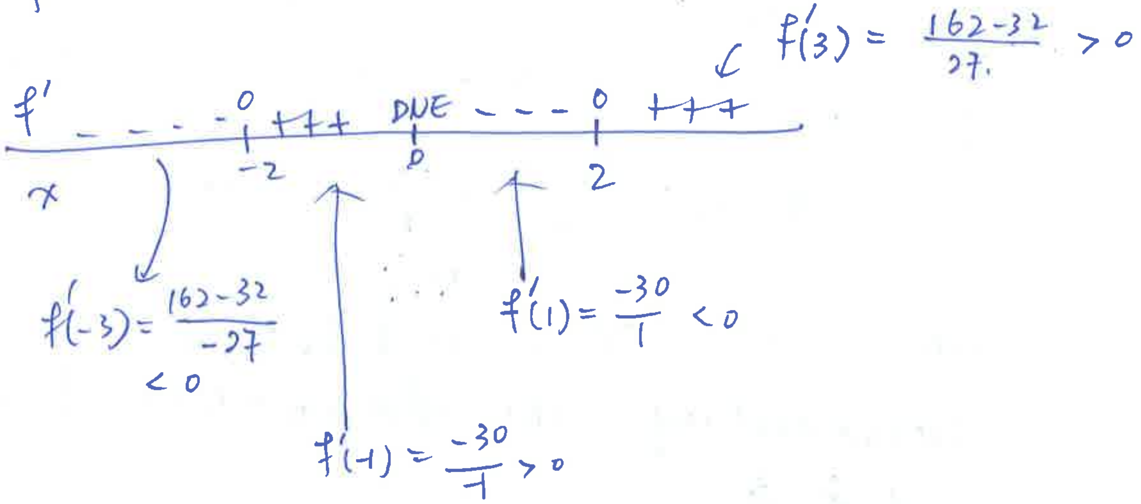
$\therefore \left(\frac{1}{4}a\sqrt{2}, \frac{-1}{4}a\sqrt{2}\right), \left(\frac{-1}{4}a\sqrt{2}, \frac{1}{4}a\sqrt{2}\right)$

$m = -1 = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} \Rightarrow y_1 = x_1 \Rightarrow x_1 = \pm\frac{1}{4}a\sqrt{2}$

$\therefore \left(\frac{1}{4}a\sqrt{2}, \frac{1}{4}a\sqrt{2}\right), \left(\frac{-1}{4}a\sqrt{2}, \frac{-1}{4}a\sqrt{2}\right)$

(16) $f(x) = x^2 + \frac{16}{x^2}$ $\quad f'(x) = 2x - \frac{32}{x^3} = \frac{2x^4 - 32}{x^3}$

$f' = 0$ when $x = \pm 2$

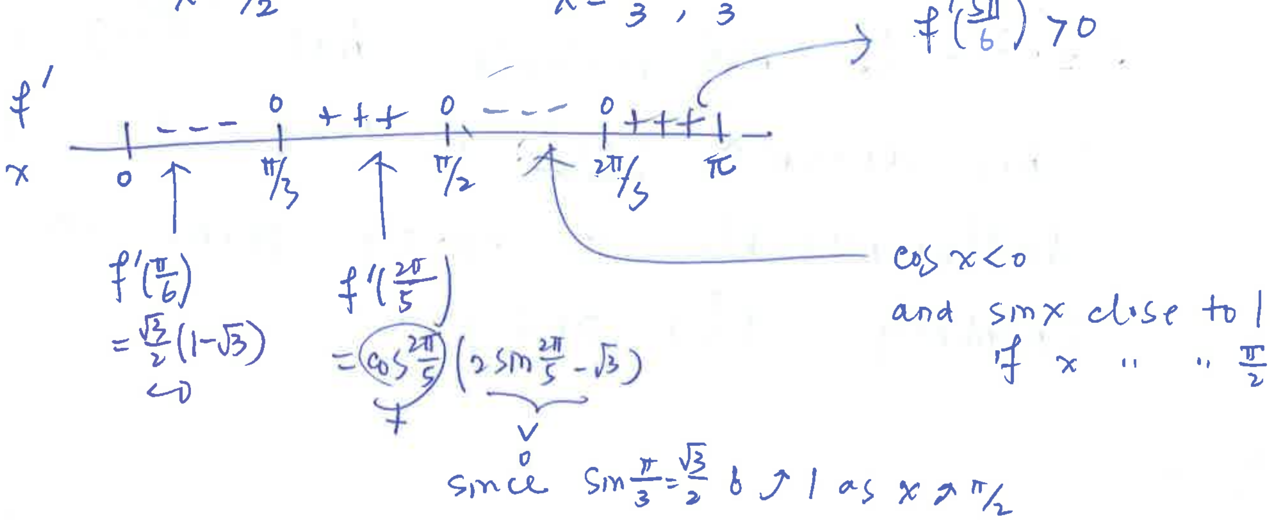


$\therefore$ f increases at $[-2, 0]$ & $(2, \infty)$
decreases " $(-\infty, -2]$ & $[0, 2]$

(24) $f(x) = \sin^2 x - \sqrt{3}\sin x$; $x \in [0, \pi]$

$f'(x) = 2\sin x \cos x - \sqrt{3}\cos x = \cos x\,(2\sin x - \sqrt{3})$

$f' = 0 \Rightarrow \underbrace{\cos x = 0}_{x = \pi/2}$ OR $\underbrace{\sin x = \frac{\sqrt{3}}{2}}_{x = \frac{\pi}{3},\ \frac{2\pi}{3}}$



f increases at $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ & $\left[\frac{2\pi}{3}, \pi\right]$

decreases at $\left[0, \frac{\pi}{3}\right]$ & $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$

HW6 More Solutions

§4-1

(14) NO! $f(1) = f(2) = f(3) = 1$

$\Rightarrow$ there is $c_1 \in (1,2)$ & $c_2 \in (2,3)$ such that

$$f'(c_1) = \frac{f(2)-f(1)}{2-1} = 0 \quad \text{and} \quad f'(c_2) = \frac{f(3)-f(2)}{3-2} = 0$$

(By MVT.)

since $c_2 \in (2,3)$, $c_2 \neq -1, \frac{3}{4}, \frac{3}{2}$

contradicting the condition that $f' = 0$ only at $-1, \frac{3}{4}, \frac{3}{2}$.

(20) $a = -1$, $b = 1$ & $f(x) = \frac{1}{x}$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{1+1}{2} = 1$$

while $f'(x) = -\frac{1}{x^2} < 0$ For all $x \neq 0$

$\therefore$ It's not possible that $f'(x) = 1$ anywhere.

This doesn't violate MVT since f is not differentiable on every point on $(-1,1)$

(indeed, $f'(0)$ DNE)

§4-2

(6) $f(x) = x(x+1)(x+2)$

$$\begin{aligned} f'(x) &= (x+1)(x+2) + x(x+2) + x(x+1) \\ &= (1+1+1)x^2 + (3+2+1)x + 2 \\ &= 3x^2 + 6x + 2 \end{aligned}$$

$f' = 0$ at $x = \frac{-3 \pm \sqrt{3}}{3}$

f': $+++$ | $----$ | $+++$

x: $\frac{-3-\sqrt{3}}{3}$, $\frac{-3+\sqrt{3}}{3}$

f increasing when $x \notin \left(\frac{-3-\sqrt{3}}{3}, \frac{-3+\sqrt{3}}{3}\right)$

decreasing when $x \in \left(\frac{-3-\sqrt{3}}{3}, \frac{-3+\sqrt{3}}{3}\right)$

//

§4-2

(56)

(a) consider $h(x) = f(x) - g(x)$

$h(0) = f(0) - g(0) = 0$

& $h'(x) = f'(x) - g'(x) > 0$ For all $x \in (0,c)$

(since $f' > g'$ on $(0,c)$)

$\therefore$ h is an increasing function on $[0,c]$

$\Rightarrow h(x) > h(0)$ For all x in $(0,c)$

(since $x > 0$)

$$\underset{f(x)-g(x)}{\overset{\shortparallel}{h(x)}} > \underset{0}{\overset{\shortparallel}{h(0)}}$$

$\Rightarrow f(x) > g(x)$ " " " "

#

(b) Again let $h(x) = f(x) - g(x)$ on $[-c, 0]$

$h(0) = f(0) - g(0) = 0$ & $h' > 0$ (h increases on $(-c, 0)$)

& $h(x) < h(0)$ on $(-c, 0)$

since $x < 0$.

$\Rightarrow f(x) - g(x) < 0 \Rightarrow f(x) < g(x)$

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