Hw7

§4-3: 6, 18, 24, 31, 38,

§4-4: 6, 14, 22, 28, 32, 34,

§4-6: 2, 4, 10, 14, 28, 42,

§4-3

Find the critical points and the local extreme values.

6.

$$f(x) = (1-x)^2 \cdot (1+x)$$

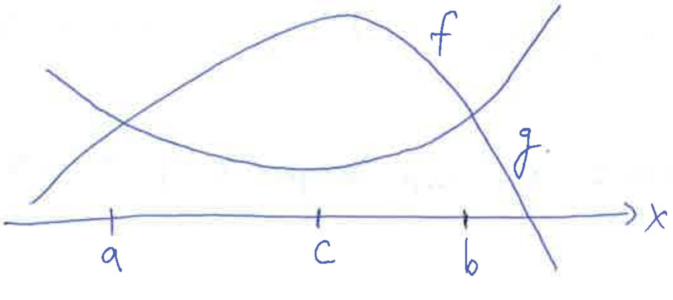
18.

$$f(x) = \frac{1}{x+1} - \frac{1}{x-2}$$

24.

$$f(x) = x + \cos 2x \ , \quad 0 < x < \pi.$$

31. Let f and g be the differentiable functions, with graphs shown below. The point c is the point in the interval $[a,b]$ where the vertical separation between two curves is greatest. Show that the line tangent to the graph of f at $x=c$ is parallel to the line tangent to the graph of g at $x=c$.



38. Suppose that $f(x) = Ax^2 + Bx + C$ has a local minimum at $x = 2$ and the graph passes through the points $(-1, 3)$ and $(3, -1)$, Find A, B, C,

§4-4

Find the critical points. Then find and classify all the extreme values,

6. $f(x) = x + \frac{1}{x^2}$

14. $f(x) = x \cdot \sqrt{4 - x^2}$

22. $f(x) = \sin 2x - x$, $0 \leq x \leq \pi$.

28.

$$f(x) = \begin{cases} 2 - 2x - x^2 , & -2 \leq x \leq 0 \\ |x - 2| , & 0 < x < 3 \\ \frac{1}{3}(x-2)^3 , & 3 \leq x \leq 4 \end{cases}$$

Sketch the graph of an everywhere differentiable function that satisfies the given conditions. If you find that the conditions are contradictory and therefore no such function exists, explain your reasoning.

32. $f(0) = 1$ the absolute minimum, local maximum at 4, local minimum at 7, no absolute maximum.

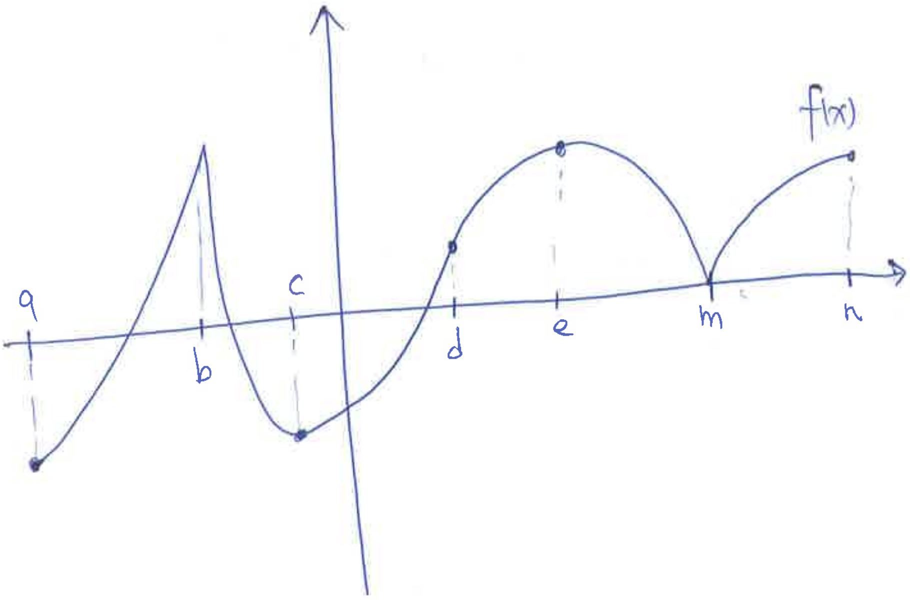
34. $f'(x) = 0$ at each integer x ; f has no extreme values.

§ 4-6.

2. The graph of a function f is given in the figure.

(a) Determine the intervals on which f increase and the intervals on which f decreases;

(b) determine the intervals on which the graph of f is concave up, the intervals on which the graph is concave down, and give the x-coordinate of each point of inflection.



4. A function f is continuous on $[-4, 4]$ and twice differentiable on $(-4, 4)$. Some information on f, f', and f'' is tabulated below:

x	$(-4,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,4)$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$-$
$f''(x)$	$-$	$-$	$-$	0	$+$	0	$-$

(a) Given the x-coordinates of the local maxima and minima of f

(b) Given the x-coordinates of the points of inflection of the graph of f.

(c) Given that $f(0)=0$, sketch a possible graph for f.

☆ Describle the concavity of the graph and find the points of inflection (if any).

10. $f(x)=x^3(1-x)$

14. $f(x)=\dfrac{6x}{x^2+1}$

38, Find

(a) the intervals on which f increases and the intervals on which f decreases;

(b) the local maxima and the local minima;

(c) the intervals on which the graph is concave up and the intervals on which the graph is concave down;

(d) the points of inflection; sketch the graph of f.

$$f(x) = 3x^4 + 4x^3 + 1$$

42, Determine A and B so that the curve $y = Ax^{1/2} + Bx^{-1/2}$ has a point of inflection at $(1, 4)$.

§ 4-3

6.

$$f(x) = (1-x)^2 \cdot (1+x)$$

$$f'(x) = 2(1-x)\cdot(-1)\cdot(1+x) + (1-x)^2 \cdot 1$$

$$= -2(1-x^2) + 1 - 2x + x^2$$

$$= 3x^2 - 2x - 1 = (3x+1)(x-1)$$

Let $f'(x) = 0 \Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = 1$ or $\frac{-1}{3}$ (critical point)

$\Rightarrow f(1) = 0$, $f(\frac{-1}{3}) = \frac{16}{9} \times \frac{2}{3} = \frac{32}{27}$

local　　　local max.

$$f''(x) = 6x - 2$$

$f''(1) = 4 > 0 \Rightarrow f(1)$: local min.

$f''(\frac{-1}{3}) = -4 < 0 \Rightarrow f(\frac{-1}{3})$: local max.

18.

$$f(x) = \frac{1}{x+1} - \frac{1}{x-2} = \frac{(x-2)-(x+1)}{(x+1)(x-2)} = \frac{-3}{x^2-x-2}, \quad x \neq 2, \quad x \neq -1$$

$$f'(x) = \frac{0-(-3)\cdot(2x-1)}{(x^2-x-2)^2} = \frac{6x-3}{(x^2-x-2)^2}$$

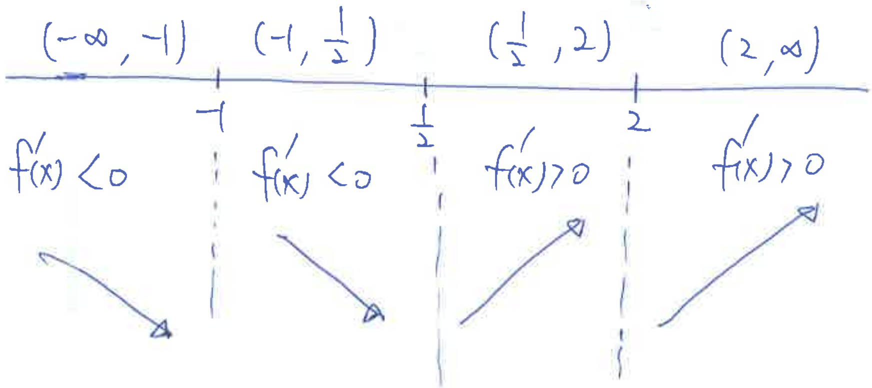
Let $f'(x) = 0$

$$\Rightarrow \frac{6x-3}{(x^2-x-2)^2} = 0$$

① $\Rightarrow 6x - 3 = 0$　　② $\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = \frac{1}{2}$　　$\Rightarrow x = 2$, $x = -1$, but $f(\frac{1}{2}) = \frac{4}{3}$, $f(2)$ and $f(-1)$ does not exist.

$\Rightarrow f'(\frac{1}{2}) = 0$　　$\Rightarrow \left\{ \begin{array}{l} f'(2) \\ f'(-1) \end{array} \right.$ does not exist. $\Rightarrow x = \frac{1}{2}$ (critical point)



$$f'(x) = \frac{6x-3}{(x^2-x-2)^2} = \frac{6x-3}{(x+1)^2(x-2)^2}$$

$\therefore f(\frac{1}{2}) = \frac{3}{4}$ (local min.)

(local extreme value)

24.

$f(x) = x + \cos 2x, \quad 0 < x < \pi.$

$f'(x) = 1 - 2\sin 2x$

Let $f'(x) = 0$

$1 - 2\cdot\sin 2x = 0$

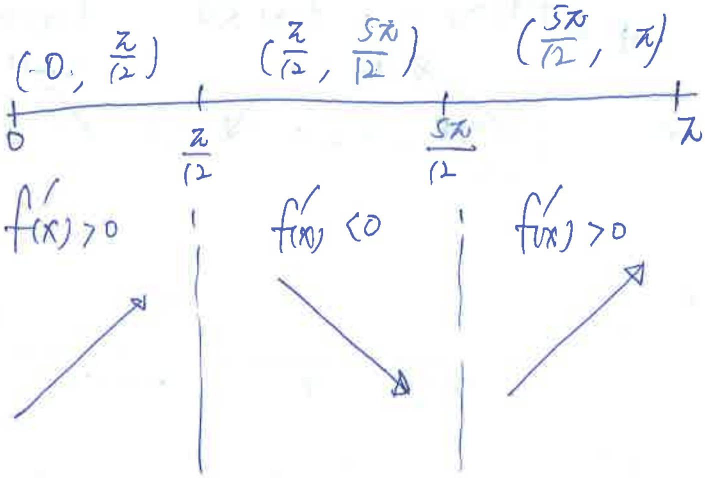
$\sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

$\Rightarrow x = \frac{\pi}{12}, \frac{5}{12}\pi$

(critical points.)



(local max)

$\therefore f(\frac{\pi}{12}) = \frac{\pi}{12} + \cos\frac{\pi}{6} = \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

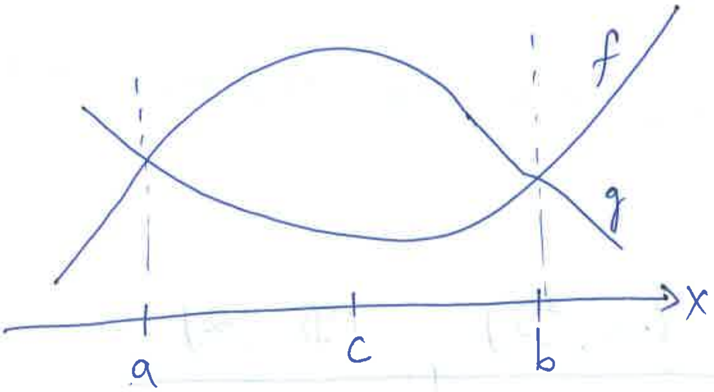
$f(\frac{5\pi}{12}) = \frac{5\pi}{12} + \cos\frac{5\pi}{6} = \frac{13\pi}{12} - \frac{\sqrt{3}}{2}$

(local extreme value)

(local min)

(3.

31.



Let $h(x) = f(x) - g(x)$, $x \in [a,b]$,

since the point c is the point in the interval [a,b] where the vertical separation between two curves is greatest.

$\Rightarrow h(x)$ has a maximum at c

$\Rightarrow h'(c) = 0$

$\forall\ h'(x) = f'(x) - g'(x)$

$\Rightarrow h'(c) = 0 = f'(c) - g'(c) \Rightarrow f'(c) = g'(c)$

Then the lines tangent to the graphs of f and g are parallel at c. ■

38.

$f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

f(2) is a local minimum

$\Rightarrow$ $x=2$ is critical point

$\Rightarrow$ $f'(2) = 0$

$f'(2) = 4A + B = 0$

$B = -4A$

$f(-1) = 3$ and $f(3) = -1$

$$\Rightarrow \begin{cases} A - B + C = 3 \\ 9A + 3B + C = -1 \\ B = -4A \end{cases} \Rightarrow \begin{cases} 5A + C = 3 \\ -3A + C = -1 \end{cases}$$

$\Rightarrow A = \frac{1}{2}$

$B = -2$

$C = \frac{1}{2}$

§4-4

6. $f(x) = x + \frac{1}{x^2} = x + x^{-2} = \frac{x^3+1}{x^2}$, $x \neq 0$.

$f'(x) = 1 - 2x^{-3} = 1 - \frac{2}{x^3}$.

$f'(0)$ does <u>not</u> exist, but $f(0)$ does <u>not</u> exist,

Let $f'(x) = 0$

$1 - \frac{2}{x^3} = 0$

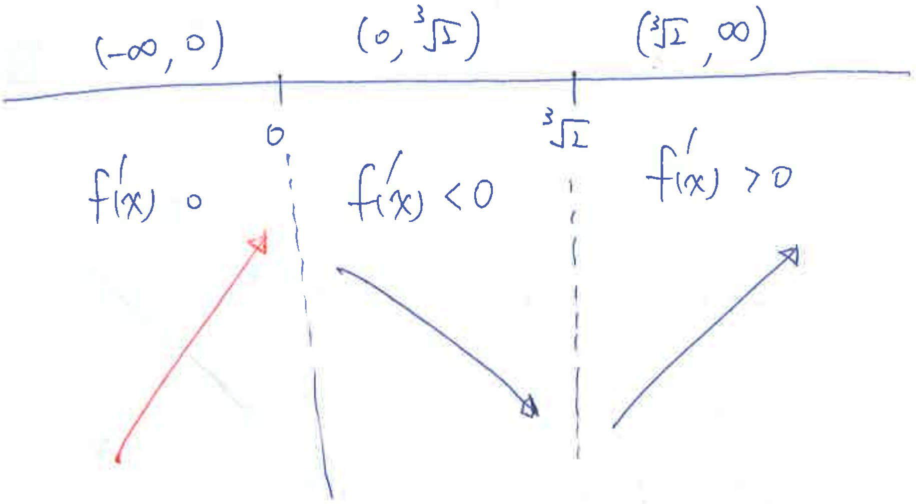
$\frac{2}{x^3} = 1$

$x^3 = 2$

$x = \sqrt[3]{2}$ (critical point)

$f(\sqrt[3]{2}) = \sqrt[3]{2} + \frac{1}{2^{2/3}} = 2^{1/3} + 2^{-\frac{2}{3}}$

(local min)



§4-4

14. $f(x) = x \cdot \sqrt{4-x^2}$, $-2 \le x \le 2$.

$$f'(x) = \sqrt{4-x^2} + x \cdot \frac{1}{2} \cdot \frac{-2x}{\sqrt{4-x^2}} = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

where $-2 < x < 2$.

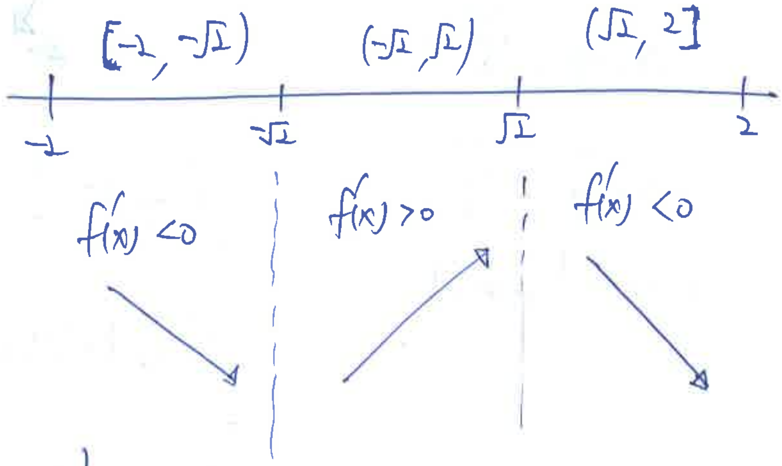
Let $f'(x) = 0$

$$\frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

$\Rightarrow$ $4-2x^2 = 0$

$\Rightarrow$ $x^2 = 2$

$\Rightarrow$ $x = \pm\sqrt{2}$ (critical point)



$f(\sqrt{2}) = 2$ (local max, absolute max)

$f(-\sqrt{2}) = -2$ (local min, absolute min)

$f(2) = 0$ (endpoint min)

$f(-2) = 0$ (endpoint max)

22.

$f(x) = \sin 2x - x$, $0 \le x \le \pi$.

$f'(x) = 2\cos 2x - 1$

Let $f'(x) = 0$

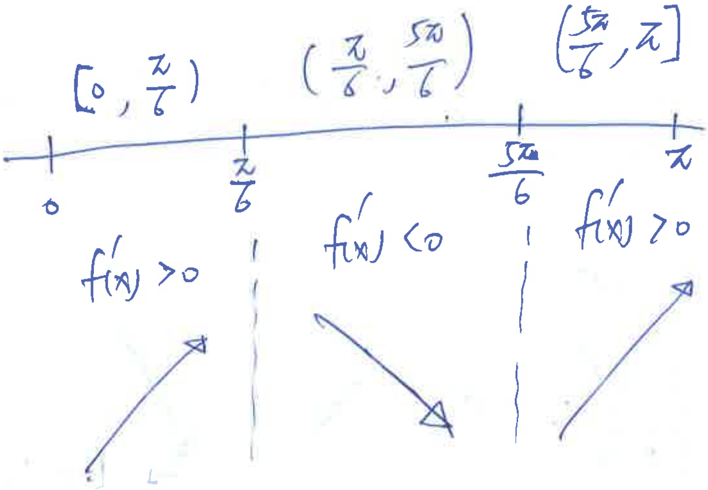
$2\cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2}$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

(critical points)



$f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ (local max, absolute max)

$f(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2} - \frac{5\pi}{6}$ (local min, absolute min)

$f(0) = 0$ (endpoint min)

$f(\pi) = -\pi$ (endpoint max)

§4-4 ✓

28,

$$f'(x) = \begin{cases} -2-2x, & -2<x<0 \\ -1, & 0<x<2 \\ 1, & 2<x<3 \\ (x-2)^2, & 3<x<4 \end{cases}$$

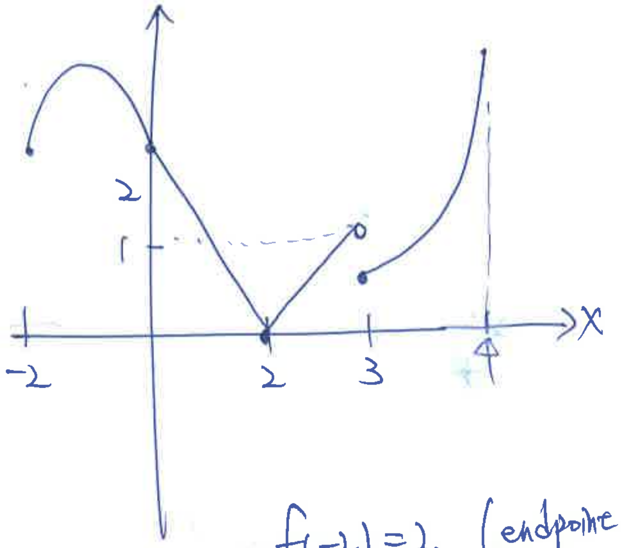
令 $f'(x) = 0$

$\Rightarrow x = -1$

and $f'(0)$ does not exist

$f'(3)$ does not exist,

$f'(2)$ does not exist,



$x = -1, 0, 2, 3$ (critical point)

$f(-2) = 2$ (endpoint min)

$f(-1) = 3$ (local max, absolute max)

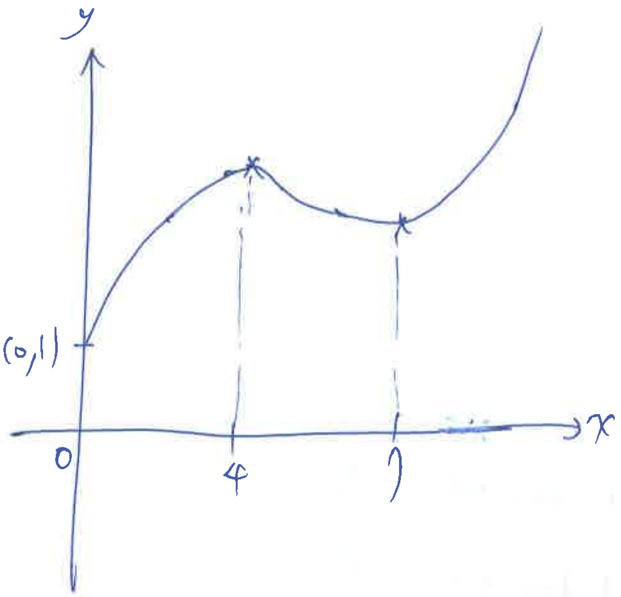
$f(0) = \underline{\underline{\text{not}}}$ an extreme value

$f(2) = 0$ (local min, absolute min)

$f(3) = \frac{1}{3}$ (local min)

$f(4) = \frac{8}{3}$ (endpoint max)

32, graph:



34,

$f(x) = x - \frac{1}{2\pi} \sin(2\pi x)$

$f'(x) = 1 - \cos(2\pi x)$

Let $f'(x) = 0$

$\Rightarrow \cos(2\pi x) = 1$

$\Rightarrow x = \cdots, -2, -1, 0, 1, 2, 3, \cdots\cdots$ (integers) (critical points)

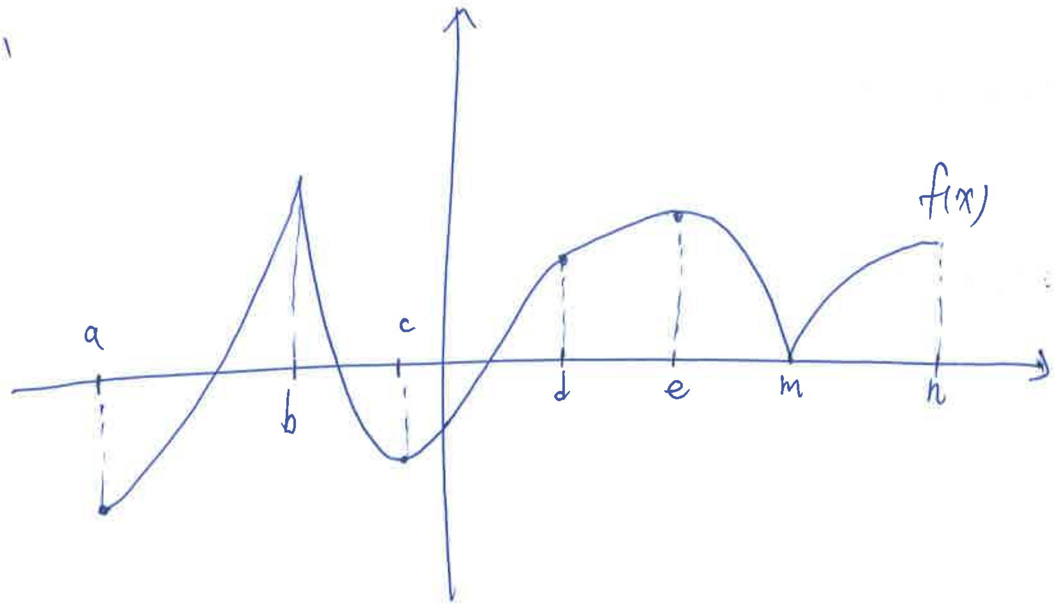
$f''(x) = \sin(2\pi x) \cdot 2\pi$

$\Rightarrow f''(n) = \sin(2\pi n) \cdot 2\pi = 0$, n: integers

$\Rightarrow$ f has no extreme values.

§4-6.

2.



(a) Increasing on [a,b], [c, e], [m,n]

decreasing on [b, c], [e, m]

(b) Concave up on (a,b), (b,d)

Concave down on (d, m), (m,n)

inflection : $x=d$

§4-6

4.

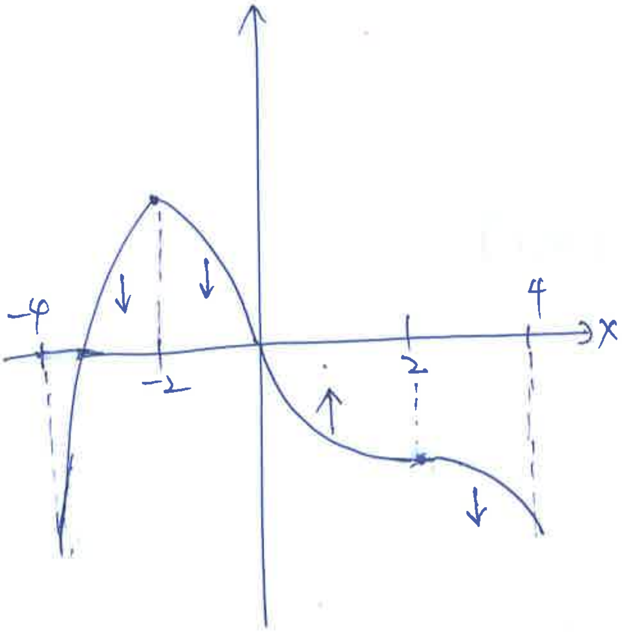
(a)

$\underline{\underline{x=-2}}$ local max , $\underline{\underline{\text{no}}}$ local min.

(b)

inflection $x=0, 2$

(c)



10. $f(x) = x^3 \cdot (1-x)$

$f'(x) = 3x^2(1-x) + x^3 \cdot (-1) = 3x^2 - 4x^3$

$f''(x) = 6x - 12x^2 = 6x(1-2x)$

Let $f''(x) = 0 \Rightarrow x = 0 , \frac{1}{2}$

$$\begin{cases} f(0) = 0 \\ f(\frac{1}{2}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16} \end{cases}$$

$$\begin{cases} f'(0) = 0 \\ f'(\frac{1}{2}) = \frac{3}{4} - \frac{4}{8} = \frac{2}{8} = \frac{1}{4} \end{cases}$$

$(-\infty, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
↓	↑	↓

0 $\quad \frac{1}{2}$

$\left\{\right.$ Concave down : $(-\infty, 0)$, $(\frac{1}{2}, \infty)$

concave up : $(0, \frac{1}{2})$

Inflection : $(0, 0)$

$(\frac{1}{2}, \frac{1}{16})$

✓

14.

$$f(x) = \frac{6x}{x^2+1}$$

$$f'(x) = \frac{(x^2+1)\cdot 6 - 6x\cdot 2x}{(x^2+1)^2} = \frac{6+6x^2-12x^2}{(x^2+1)^2} = \frac{6-6x^2}{(1+x^2)^2}$$

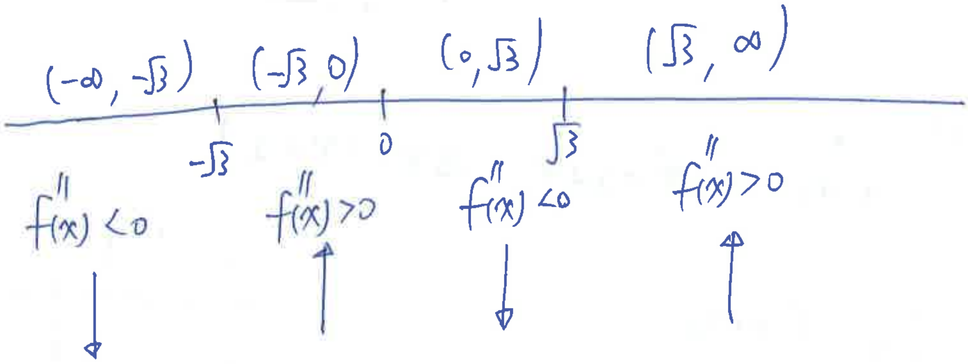
$$f''(x) = \frac{(1+x^2)^{\cancel{2}}\cdot(-12x) - (6-6x^2)\cdot 2\cdot \cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}\,3}} = \frac{-12x-12x^3-24x+24x^3}{(1+x^2)^3}$$

$$= \frac{12x^3-36x}{(1+x^2)^3} = \frac{12x(x^2-3)}{(1+x^2)^3}$$

Let $f''(x) = 0$

$\Rightarrow 12x(x^2-3) = 0$

$\Rightarrow x = 0, \pm\sqrt{3}$



$f(0) = 0$ $\quad f(\sqrt{3}) = \frac{6\sqrt{3}}{1+3} = \frac{3\sqrt{3}}{2}$ $\quad f(-\sqrt{3}) = \frac{-6\sqrt{3}}{1+3} = \frac{-3\sqrt{3}}{2}$

$f'(\sqrt{3}) = \frac{-12}{16}$ $\quad f'(-\sqrt{3}) = \frac{-12}{16}$

concave down : $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

Concave up : $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$

Inflections : $(-\sqrt{3}, -\frac{3}{2}\sqrt{3})$, $(\sqrt{3}, \frac{3\sqrt{3}}{2})$

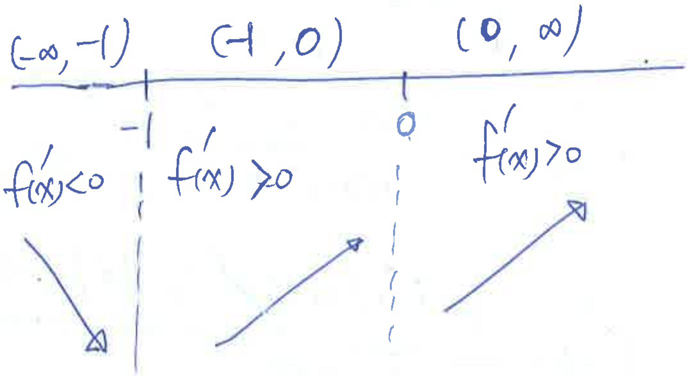
28.

(a) $f(x)=3x^4+4x^3+1$

$f'(x)=12x^3+12x^2=12x^2(x+1)$

Let $f'(x)=0$

$\Rightarrow x=-1, 0$ (critical points)



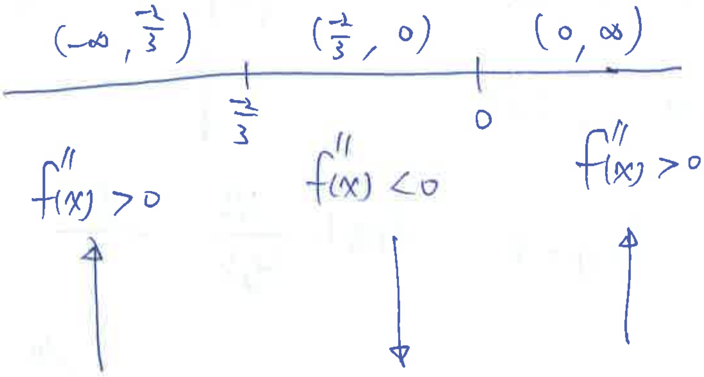
Increasing: $[-1,\infty)$ decreasing: $(-\infty,-1]$

(b) $f(-1)=0$ (local min), no local max.

(c) $f''(x)=36x^2+24x=12x(3x+2)$.

Let $f''(x)=0$

$\Rightarrow x=0, \frac{-2}{3}$



Concave down: $(\frac{-2}{3},0)$

Concave up: $(-\infty,\frac{-2}{3})$, $(0,\infty)$

(d) $f(0)=1$ $f'(0)=0$

$f(\frac{-2}{3})=\frac{11}{27}$ $f'(\frac{-2}{3})=12\times\frac{4}{9}\times\frac{1}{3}=\frac{16}{9}$

Inflection: $(\frac{-2}{3},\frac{11}{27})$, $(0,1)$

42.

$$y = A\cdot x^{\frac{1}{2}} + B\cdot x^{-\frac{1}{2}} \quad —①$$

$$y' = \frac{1}{2}A\cdot x^{-\frac{1}{2}} - \frac{1}{2}B\cdot x^{-\frac{3}{2}} \quad —②$$

$$y'' = \frac{-1}{4}A\cdot x^{-\frac{3}{2}} + \frac{3}{4}B\cdot x^{-\frac{5}{2}} \quad —③$$

inflection = (1, 4)

$$\begin{cases} x=1,\ y=4 \text{ 代入 } ① \Rightarrow 4 = A+B \\ x=1 \text{ 代入 } y''=0 \Rightarrow 0 = \frac{-1}{4}A + \frac{3}{4}B \end{cases} \Rightarrow \begin{cases} A+B=4 \\ -A+3B=0 \end{cases}$$

$$\Rightarrow \underline{A=3} \quad \underline{B=1}$$