HW 8:

§4.5: 12, 15, 22, 33, 59, 62

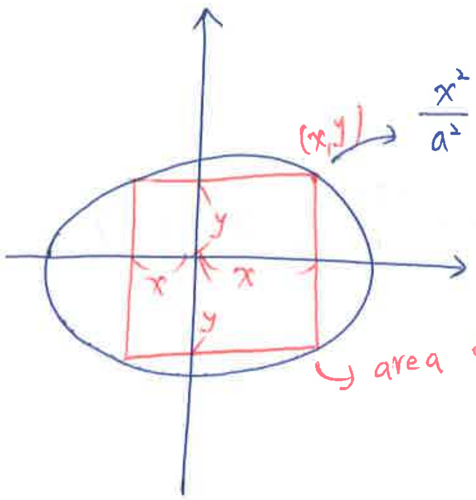
§4.7: 2, 8, 14, 26, 34

§4.8: 6, 14, 30, 51

§4.5

※ 22.

Find the maximal possible area for a rectangle inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$ $\quad y^2 = b^2 - \frac{b^2}{a^2}x^2 = \frac{b^2}{a^2}(a^2 - x^2)$

↳ area of rectangle is $(2x)\cdot(2y) = 4xy = 4x \cdot \frac{b}{a} \cdot \sqrt{a^2 - x^2}$

$0 \le x \le a$

$0 \le y \le b$

Let $f(x) = \frac{4b}{a} x \cdot \sqrt{a^2 - x^2}$, $\quad 0 \le x \le a$

$$f'(x) = \frac{4b}{a} \cdot \sqrt{a^2 - x^2} + \frac{4b}{a} x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{a^2 - x^2}} \cdot (-2x) = \frac{4b}{a}\sqrt{a^2 - x^2} - \frac{4b}{a} \cdot \frac{x^2}{\sqrt{a^2 - x^2}}$$

$$= \frac{4b}{a} \cdot \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

$f'(x) = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$, $x \ne a$ (if $x = a$, then this rectangle does not exist ($\because y = 0$))

$\Rightarrow y = \frac{a}{\sqrt{2}}$

$f\left(\frac{a}{\sqrt{2}}\right) = \frac{4b}{a} \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{a^2 - \frac{a^2}{2}} = \frac{4b}{a} \cdot \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \boxed{2ab}$ (maximal possible area) ※

59. A truck is to be driven 300 miles on a freeway at a constant speed of v miles per hour. Speed laws require that $35 \le v \le 70$. Assume that the fuel costs \$ 2.60 per gallon and is consumed at the rate of $1 + (\frac{1}{400})v^2$ gallons per hour. Given that the driver's wages are \$ 20 per hour, at what speed should the truck be driven to minimize the truck owner's expenses?

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{300}{v} \text{ (hour)}$$

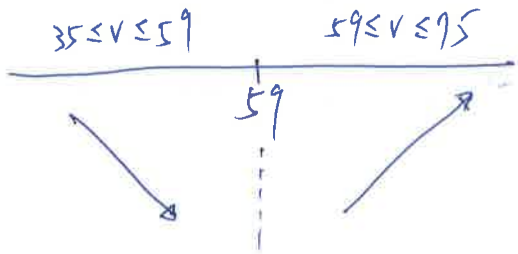
$$f(v) = 2.6 \times \left(1 + \frac{1}{400}v^2\right) \cdot \frac{300}{v} + 20 \cdot \frac{300}{v}$$

$$= \frac{780}{v} + \frac{7.8}{4}v + \frac{6000}{v} = 1.95v + \frac{6780}{v}, \quad 35 \le v \le 70.$$

$$f'(v) = 1.95 - \frac{6780}{v^2}$$

$$f'(v) = 0 \Rightarrow 1.95 = \frac{6780}{v^2}$$

$$v^2 = 3476.92\cdots \quad v = 58.9\cdots \approx \boxed{59}$$



local minimize !!

§4.7

※ 14. Find the vertical and horizontal asymptotes:

$$f(x) = \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}} - 4}$$

① Let $x^{\frac{2}{3}} - 4 = 0 \Rightarrow x^{\frac{2}{3}} = 4 \Rightarrow x = 4^{\frac{3}{2}} = 8$

$$\lim_{x \to 8} \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}} - 4} = +\infty$$ ∴ vertical asymptotes: $x = 8$

②
$$\lim_{x \to +\infty} \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}} - 4} = \lim_{x \to +\infty} \frac{x^{-\frac{1}{3}}}{1 - \frac{4}{x^{2/3}}} = \frac{0}{1-0} = 0$$

$$\because \lim_{x \to +\infty} x^{-\frac{1}{3}} = \lim_{x \to +\infty} \frac{1}{\sqrt[3]{x}} = 0$$

$$\because \lim_{x \to +\infty} \frac{4}{x^{2/3}} = \lim_{x \to +\infty} \frac{4}{\sqrt[3]{x^2}} = 0$$

$$\lim_{x \to -\infty} \frac{x^{\frac{1}{3}}}{x^{2/3} - 4} = \lim_{x \to -\infty} \frac{x^{-\frac{1}{3}}}{1 - \frac{4}{x^{2/3}}} = \frac{0}{1-0} = 0$$

$$\because \lim_{x \to -\infty} x^{-\frac{1}{3}} = \lim_{x \to -\infty} \frac{1}{\sqrt[3]{x}} = 0$$

$$\because \lim_{x \to -\infty} \frac{4}{x^{2/3}} = \lim_{x \to -\infty} \frac{4}{\sqrt[3]{x^2}} = 0$$

∴ horizontal asymptotes: $y = 0$

§ 4.8.

Sketch the graph of the function using the approach presented in this section.

14. $f(x) = \frac{1}{4}x - \sqrt{x}$, $0 \le x \le 9$.

14.

$f'(x) = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{4} - \frac{1}{2}\cdot x^{-\frac{1}{2}}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{4}\cdot\frac{1}{x\sqrt{x}}$

$f'(x) = 0 \Rightarrow \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$\Rightarrow \frac{1}{\sqrt{x}} = \frac{1}{2}$

$\Rightarrow x = 4$

$f(0) = 0$

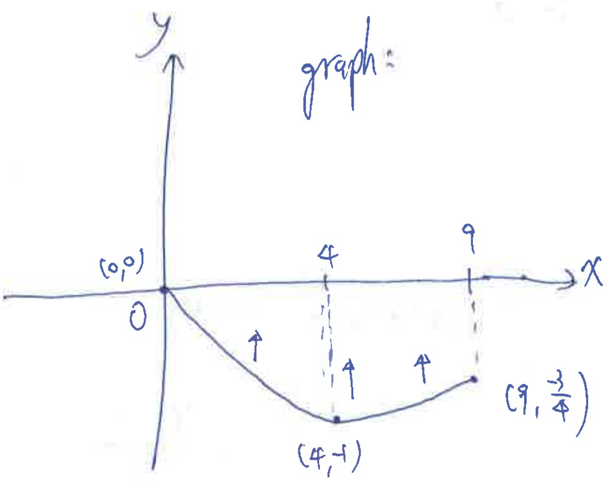
$f(4) = 1 - 2 = -1$

$f(9) = \frac{9}{4} - 3 = \frac{-3}{4}$

0 — $f'(x) < 0$ ↘ — 4 — $f'(x) > 0$ ↗ — 9

$f''(x)$: 0 — $f''(x) > 0$ ↑ — 9

graph:



30. $f(x) = \cos^3 x + 6\cos x$, $0 \le x \le \pi$.

$f'(x) = 3\cos^2 x\cdot(-\sin x) + 6\cdot(-\sin x)$

$= -3\sin x(\cos^2 x + 2)$

$f''(x) = -3\cos x(\cos^2 x + 2) + (-3\sin x)\cdot(2\cos x\cdot(-\sin x))$

$= -3\cos^3 x - 6\cos x + 6\sin^2 x\cdot\cos x$

$= -3\cos^3 x - 6\cos x\cdot(1 - \sin^2 x)$

$= -3\cos^3 x - 6\cos x\cdot\cos^2 x = -9\cos^3 x$

$f'(x)$: 0 — $f'(x) < 0$ ↘ — π

$f(0) = 7$

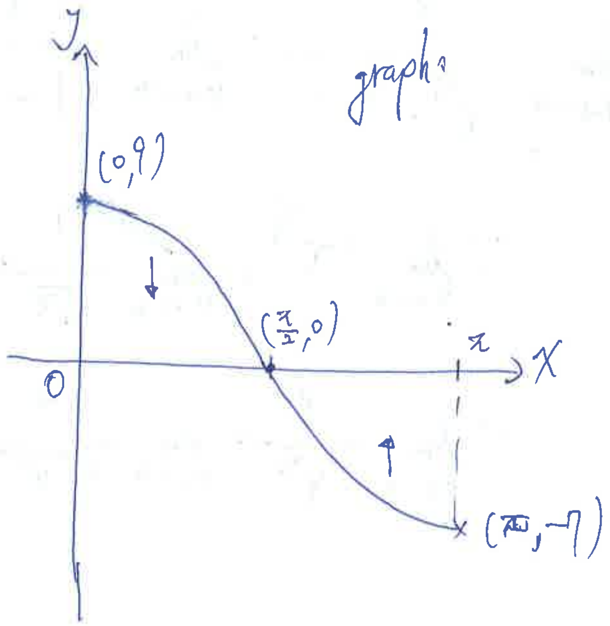
$f(\frac{\pi}{2}) = 0$

$f(\pi) = -7$

Let $f''(x) = 0 \Rightarrow x = \frac{\pi}{2}$

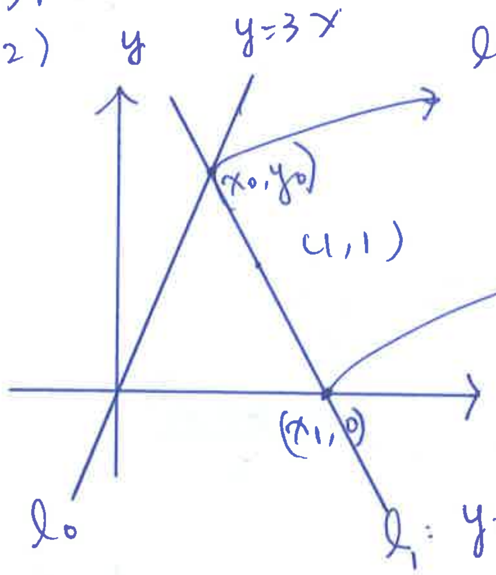
$f''(x)$: 0 — $f''(x) < 0$ ↓ — $\frac{\pi}{2}$ — $f''(x) > 0$ ↑ — π

graph:



§4-5

(12)



$l_1 = l_2$: $3x_0 = mx + 1 - m$

$\Rightarrow x_0 = \frac{1-m}{3-m} \rightsquigarrow y_0 = \frac{3-3m}{3-m}$

$0 = mx_1 + 1 - m$, $\quad x_1 = \frac{m-1}{m}$

since l_1 passes thru. (1,1)

$$\text{Area of } \Delta = A(m) = \frac{1}{2} \cdot x_1 \cdot y_0 = \frac{1}{2} \frac{m-1}{m} \frac{3-3m}{3-m}$$

$$= \frac{3}{2} \frac{(m-1)^2}{m^2 - 3m}$$

$$\rightsquigarrow A'(m) = \frac{3}{2} \frac{2(m-1)(m^2-3m) - (2m-3)(m-1)^2}{(m^2-3m)^2}$$

$$= -\frac{3}{2} \frac{(m+3)(m-1)}{(m^2-3m)^2}$$

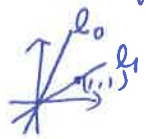
$A'(m) = 0$ at $m = -3, 1$, and undefined at $0, 3$

$A'(m)$	− − −	+ + +	?	+ + +	− − −	?
m		-3	0		1	3

suspect: $-3, 0, 1, 3$

$A(-3) = \frac{8}{6} = \frac{4}{3}$

$A(0) =$ undefined (no Δ formed)

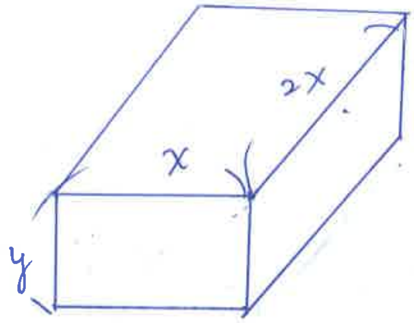
$A(1) = 0$ (no $\Delta \rightarrow$ )

$A(3) \Rightarrow$ $(l_0 = l_1)$

$\therefore$ max. area occurs when $m = -3$

where $A(-3) = \frac{4}{3}$ //

(15)



Total surface area = 100 (in²)

$2 \cdot 2x^2 + 2 \cdot xy + 2 \cdot 2xy = 100$

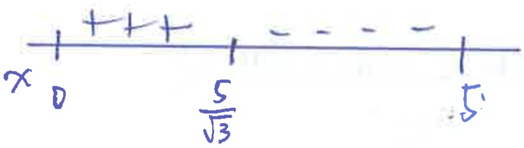
$= 4x^2 + 6xy$

$\Rightarrow y = \dfrac{100 - 4x^2}{6x}$

$V = 2x^2 y = 2x^2 \dfrac{100 - 4x^2}{6x} = \dfrac{100x - 4x^3}{3}$

$V'(x) = \dfrac{100 - 12x^2}{3}$; $V' = 0$ when $x = \pm \frac{5}{\sqrt{3}}$

no negative length



Moreover,

$x^2 < 100$

since we don't have more than 100 in² of paper.

$\Rightarrow 0 \le x \le 5.$

$V(0) = 0$

$V(5) = 0$

$V\left(\frac{5}{\sqrt{3}}\right) = \dfrac{1000}{9\sqrt{3}}$ ← max. volume

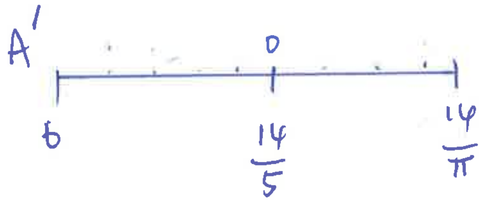
(33)

$2\pi r + 4x = 28$

$\rightarrow \quad x = \dfrac{28 - 2\pi r}{4} = 7 - \frac{1}{2}\pi r.$

$\therefore \; A(r) = \pi r^2 + x^2 = \pi r^2 + \left(7 - \frac{1}{2}\pi r\right)^2 = \frac{5}{4}\pi r^2 - 7\pi r + 49$

$A'(r) = \frac{5}{2}\pi r - 7\pi$, $A' = 0$ at $r = \frac{14}{5}$



$2\pi r \leq 28$

$r \leq \frac{14}{\pi}$

$A(0) = 49$

$A\left(\frac{14}{5}\right) = 49 - \frac{49\pi}{5} \sim 18$

$A\left(\frac{14}{\pi}\right) = \pi\left(\frac{14}{\pi}\right)^2 = \frac{196}{\pi} \sim 62.4$

$\therefore$ A max. at $r = 0$

ie. all string used to construct square of side 7.

and A minimized at $r = \frac{14}{5}$

w/ $A\left(\frac{14}{5}\right) \sim 18$

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(62) See class notes.

(2) ⓐ $f(x) \to d$ as $x \to \infty$

ⓑ $f(x) \to c$ as $x \to b^+$

ⓒ vertical asymptoe

$x = b$, as $x \to b^-$ ($y \to$)

$x = a$, as $x \to a^+$ ($y \to +\infty$)

$x \to a^-$ ($y \to -\infty$)

ⓓ horizontal asymptoe

$y = d$, as $x \to +\infty$

ⓔ vertical tangent occurs at

~~$x = q$~~, and $x = p$.

ⓕ vertical cusp at $x = q$, (derivative DNE)

(8) Horizontal Asymp.

$$\lim_{x \to +\infty} \frac{\sqrt{x}}{4\sqrt{x} - x} = \lim_{x \to +\infty} \frac{\sqrt{1/x}}{4\sqrt{1/x} - 1} = 0$$

$\therefore$ $y \to 0$ as $x \to +\infty$

no asymp. as $x \to -\infty$ since f not defined on $x < 0$.

Vertical Asymp:

since $\sqrt{16} < 4$

$4\sqrt{x} - x = 0 \rightsquigarrow x = 0, 16.$

$\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to 16^-} f(x) = \lim_{x \to 16^-} \frac{1}{4 - \sqrt{x}} = +\infty$ & $\lim_{x \to 16^+} \frac{1}{4 - \sqrt{x}} = -\infty$.

(26) $f(x) = (x-5)^{\frac{7}{5}}$

$\Rightarrow f'(x) = \frac{7}{5}(x-5)^{\frac{2}{5}} = \frac{7}{5}\sqrt[5]{(x-5)^2}$, which is defined and finite everywhere.

$\therefore$ no vertical tangent nor cusp.

(34) $f(x) = \begin{cases} 1+\sqrt{-x} ; & x \le 0 \\ (4x-x^2)^{\frac{1}{3}} ; & x > 0 \end{cases}$

$f'(x) = \begin{cases} 1 - \frac{1}{2\sqrt{-x}} & x < 0 \\ \frac{1}{3}(4x-x^2)^{-\frac{2}{3}} \cdot (4-2x) ; & x > 0 \end{cases}$

$\lim_{x \to 0^+} f'(x) = -\infty$

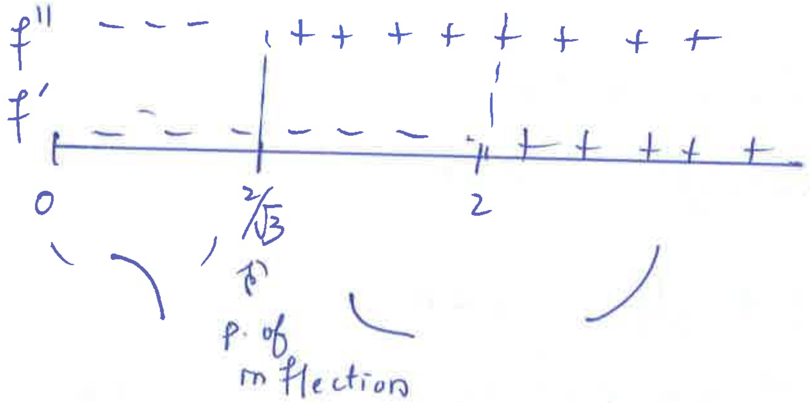
$\lim_{x \to 0^+} f'(x) = \frac{4}{3} \cdot \lim_{x \to 0^+} \frac{1}{\sqrt[3]{(4x-x^2)^2}} = +\infty$

$\therefore$ vertical cusp at $x = 0$.

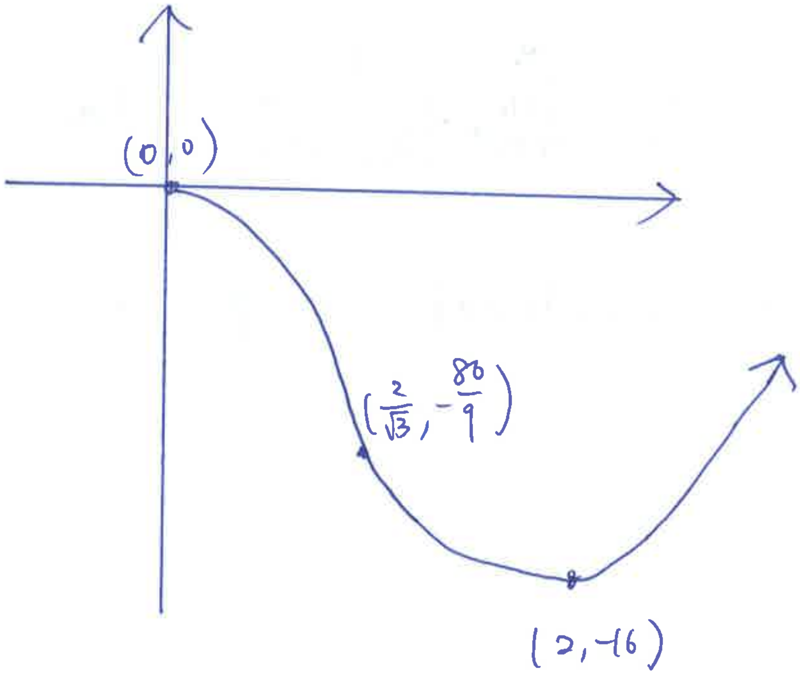
16) $f(x) = x^4 - 8x^2 \quad ; \quad x \in (0, \infty)$

$f'(x) = 4x^3 - 16x \quad ; \quad f''(x) = 12x^2 - 16$

$f' = 0$ at $x = 0, 2, \cancel{-2}$ (not in domain) $\qquad f'' = 0$ at $x = \frac{2}{\sqrt{3}}, \cancel{-\frac{2}{\sqrt{3}}}$ (not in domain)



$f(0) = 0$, $\quad f\left(\frac{2}{\sqrt{3}}\right) = -\frac{80}{9}$ $\qquad f(2) = -16.$



(51) $f(x) = 2\tan x - \sec^2 x \quad ; \quad x \in (0, \frac{\pi}{2})$

$f'(x) = 2\sec^2 x - 2\sec^2 x \tan x \quad ;$

$= 2\sec^2 x (1 - \tan x)$

$\sec^2 x \neq 0$

$f' = 0$ when $\tan x = 1$ OR $x = \frac{\pi}{4}$

$f''(x) = 4\sec^2 x \tan x - 4\sec^2 \tan^2 x - 2\sec^4 x$

$= \sec^2 x (4\tan x - 4\tan^2 x - 2\sec^2 x)$

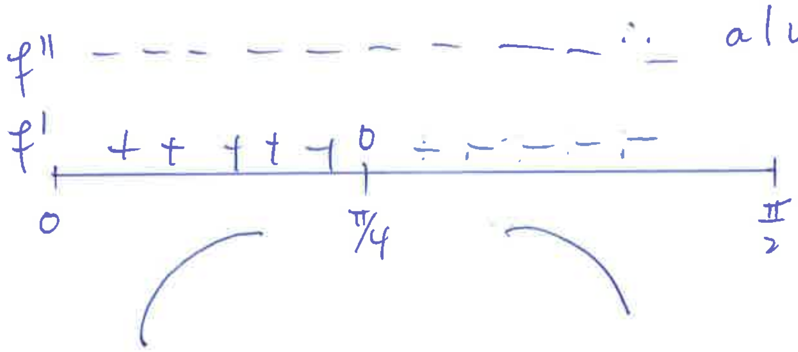
since $\sec^2 x = \tan^2 x + 1$ $\Rightarrow$ $= \sec^2 x(-2\tan^2 x + 4\tan x - 2)$

$= -2\sec^2 x(\tan^2 x - 2\tan x + 1)$

$= -2\sec^2 x(\tan x - 1)^2$

≤ 0

$\therefore$ always concave down



$f(0) = -1$

$f(\pi/4) = 2 - 2 = 0$

$f(\frac{\pi}{2}) = \infty$