XI. Change of Basis, Diagonalization

Motivation: Given want to write down an expression for

$$T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$$

$\Updownarrow$ (fixing basis)

Given $A \in Mat_{n\times n}$, compute $\underbrace{A^m = A \cdots A}_{\text{can be very troublesome}}$

computing A^m can be troublesome, but there are some special convenient cases.

eg., when A is "diagonal", i.e. $a_{ij} \neq 0 \Rightarrow i = j$

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} := \text{diag.}(\lambda_1, \dots, \lambda_n)$$

$$\rightsquigarrow \quad A^m = \text{diag.}(a_1^m, \dots, a_n^m)$$

Q: Given $T : (\mathbb{R}^n, \beta) \to (\mathbb{R}^n, \beta)$. can (& how) we find an alternative basis γ so that

$$T : (\mathbb{R}^n, \gamma) \to (\mathbb{R}^n, \gamma)$$

has matrix $[T]_\gamma^\gamma = \text{diag.}(\lambda_1, \dots, \lambda_n)$?

If we can, what is the relationship between

$$[T]_\beta^\beta \text{ and } [T]_\gamma^\gamma \text{ ?}$$

An elementary example:

on $\mathbb{R}^2$, $\beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$T : (\mathbb{R}^2, \beta) \to (\mathbb{R}^2, \beta)$ defined by

$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+3y\\2x+y\end{pmatrix}$; $[T]_\beta^\beta = \begin{pmatrix}2&3\\2&1\end{pmatrix}$.

If we choose $\gamma = \{\gamma_1, \gamma_2\}$ so that

$(\gamma_1)_\beta = \begin{pmatrix}1\\-1\end{pmatrix}$, $(\gamma_2)_\beta = \begin{pmatrix}1\\\frac{2}{3}\end{pmatrix}$

$\rightsquigarrow \quad (T(\gamma_1))_\beta = \left(T\begin{pmatrix}1\\0\end{pmatrix}\right)_\beta - \left(T\begin{pmatrix}0\\1\end{pmatrix}\right)_\beta = \begin{pmatrix}-1\\1\end{pmatrix} = -\begin{pmatrix}1\\-1\end{pmatrix}$

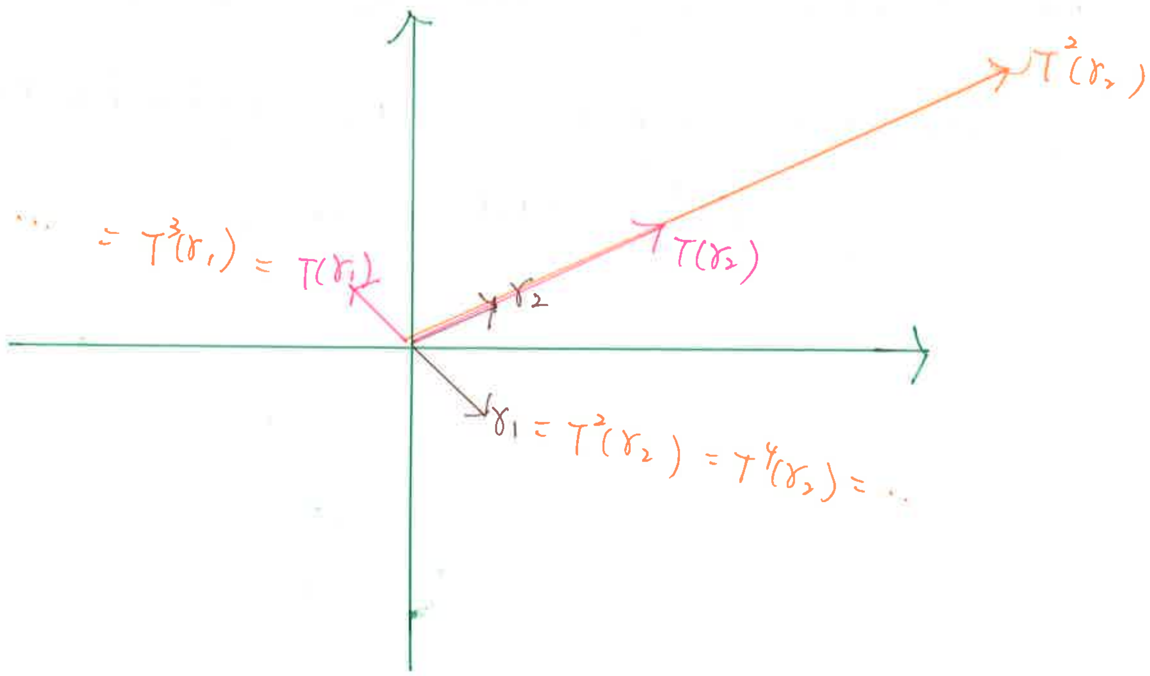
or, without coordinate, $T(\gamma_1) = -\gamma_1$.

$\rightsquigarrow \quad (T(\gamma_2))_\beta = \left(T\begin{pmatrix}1\\0\end{pmatrix}\right)_\beta + \frac{2}{3}\left(T\begin{pmatrix}0\\1\end{pmatrix}\right)_\beta = \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}2\\\frac{2}{3}\end{pmatrix} = \begin{pmatrix}4\\\frac{8}{3}\end{pmatrix}$

$= 4 \cdot \begin{pmatrix}1\\2/3\end{pmatrix}$

" " " $T(\gamma_2) = 4\gamma_2$.

We conclude: $[T]_\gamma^\gamma = \begin{pmatrix}-1&0\\0&4\end{pmatrix}$ and $([T]_\gamma^\gamma)^m = \begin{pmatrix}(-1)^m&0\\0&4^m\end{pmatrix}$



* Choice of Basis vs. Matrix Representation

Given $T: V \to W$, choose two sets of bases

$\beta = \{\beta_j\}_{j=1}^{n}$, $\beta' = \{\beta'_j\}_{j=1}^{n}$ for V and

$\gamma = \{\gamma_i\}_{i=1}^{m}$, $\gamma' = \{\gamma'_i\}_{i=1}^{m}$ for W.

What is the relationship between $[T]_\beta^\gamma$ and $[T]_{\beta'}^{\gamma'}$?

$$
\begin{tikzcd}
(V,\beta) \arrow[r, "T"] & (W,\gamma) \\
(V,\beta') \arrow[u, "I_V"] \arrow[r, "T"] & (W,\gamma') \arrow[u, "I_W"']
\end{tikzcd}
$$

有修改，請對照

As transformation, $T = I_W \circ T \circ I_V^{-1}$

As matrices $[T]_\beta^\gamma = [I_W \circ T \circ I_V^{-1}]_\beta^\gamma$

$$= [I_W]_{\gamma'}^{\gamma}\,[T]_{\beta'}^{\gamma'}\left([I_V]_{\beta'}^{\beta}\right)^{-1}$$

$$\Rightarrow [T]_{\beta'}^{\gamma'} = \left([I_W]_{\gamma'}^{\gamma}\right)^{-1} [T]_\beta^\gamma \, [I_V]_{\beta'}^{\beta} \quad \cdots (*)$$

change of coordinate matrices

加註

i.e. $\beta = \{\beta_1, \dots, \beta_n\}$, $\beta' = \{\beta'_1, \dots, \beta'_n\}$

$\Rightarrow [I_V]_{\beta'}^{\beta} = \left((\beta'_1)_\beta, \dots, (\beta'_n)_\beta\right)$; column j := coordinate representation of β'_j in basis β

上課未提，可忽略

We have the correspondence

$$\mathcal{L}(V,W) \longleftrightarrow \mathrm{Mat}_{m\times n}/\sim$$

"equivalent class"

$A \sim B \Leftrightarrow A = QBP$, an equivalent relation, since

① $A = I_m A I_n \Rightarrow A \sim A$

② $A \sim B \Rightarrow A = QBP \Rightarrow B = Q^{-1}AP^{-1} \Rightarrow B \sim A$

③ $A \sim B$ & $B \sim C \Rightarrow A = QBP$ & $B = Q'CP' \Rightarrow A = QQ'CP'P \Rightarrow A \sim C$

In particular, we discuss $V = W$ & $\beta = \gamma$. ④

$\circledast$ becomes

$$[T]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta}$$
$$= \left([I_V]_{\beta'}^{\beta}\right)^{-1} [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta}$$

Two $n \times n$ matrices A, B are called similar if

$$B = \underbrace{Q^{-1} A Q}_{A \text{ conjugated with } Q}$$

Two similar matrices represent the same linear transformation, with appropriate choices of basis:

請對照修改

i.e. $B = Q^{-1} A Q$

if $A = [T]_{\beta}^{\beta} \Rightarrow B = [T]_{\beta'}^{\beta'}$, where $Q = [I_V]_{\beta'}^{\beta}$

and we have the correspondence

$$\mathcal{L}(V, V) \longleftrightarrow \mathrm{Mat}_{n \times n}/\sim$$

$A \sim B \Leftrightarrow A$ is similar to B

Back to the elementary example.

加註 $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $\beta' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} \right\}$: clearly

β_1, β_2, β_1', β_2' in coordinates $\{e_j\}_{j=1}^n$

β_1': $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

β_2': $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(\beta_1')_{\beta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(\beta_2')_{\beta} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

修改

$$\Rightarrow Q = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix}$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix}$$

修改

and

$$\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix}}_{\text{Diagonalization of } \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}}$$

Dragonalization is a convenient tool to describe

$$T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$$

Observe: if $B = Q^{-1} A Q \Rightarrow B^m = Q^{-1} A^m Q$ (induction)

and if $B = \text{diag.}(\lambda_1, \ldots, \lambda_n) \Rightarrow B^m = \text{diag}(\lambda_1^m, \ldots, \lambda_n^m)$

$$\Rightarrow A^m = Q \, \text{diag}(\lambda_1^m, \ldots, \lambda_n^m) \, Q^{-1}.$$

In the elementary example, we have,

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \begin{pmatrix} (-1)^m & 0 \\ 0 & 4^m \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix}$$

修改

But we still need to find the $\{\gamma_j\}_{j=1}^n$ so that

$$[T]_\gamma^\gamma = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

* Diagonalization

Def: Given $T \in \mathcal{L}(V,V)$, a vector $\overset{0}{\underset{\shortparallel}{\ }} \!\!\!\!\!\! v \in V$ is an eigenvector of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. λ is called the corresponding eigenvalue.

Observe: a necessary condition for λ and v is that
$v \in \mathcal{N}(T-\lambda I)$

since $T(v) = \lambda v \Rightarrow T(v) - \lambda v = 0$
$\Rightarrow T(v) - \lambda I(v) = 0$
$\Rightarrow (T - \lambda I)(v) = 0.$

$\therefore$ for any basis β of V, $(v)_\beta \in \mathcal{N}([T-\lambda I]_\beta^\beta)$

since $v \neq 0 \Rightarrow \mathcal{N}([T-\lambda I]_\beta^\beta) \neq \{0\} \Rightarrow [T-\lambda I]_\beta^\beta$ not invertible

$\Rightarrow \det([T-\lambda I_n]_\beta^\beta) = 0$

OR $\underbrace{\det([T]_\beta^\beta - \lambda I_n) = 0}_{\text{eq. to solve for } \lambda.}$

Note: $\det([T]_\beta^\beta - \lambda I_n)$ is a polynomial in λ of degree $\leq n$

$\therefore$ There are at most n eigenvalues,

and for each λ_i, $1 \leq i \leq r \leq n$, we row reduce $[T-\lambda_i I]_\beta^\beta$ to find its $\underbrace{\text{null space}}_{\text{eigenspace for } \lambda_i}$, whose basis elements are corresponding eigenvectors v.

If we can find n independent eigenvectors $v_1, \dots, v_n$

$\Rightarrow$ they form a basis called **eigenbasis**, and $\gamma = \{v_j\}_{j=1}^n$

$$[T]_\gamma^\gamma = \operatorname{diag}(\underbrace{\lambda_1 \cdots \lambda_1}_{m_1} \; \underbrace{\lambda_2 \cdots \lambda_2}_{m_2} \cdots \cdots \underbrace{\lambda_r \cdots \lambda_r}_{m_r})$$

where m_j = nullity of $[T]_\beta^\beta - \lambda_j I_n$

and T (or the matrix $[T]_\beta^\beta$) is called **diagonalizable**

eg" $T: \mathbb{R}^3 \to \mathbb{R}^3$ w/

$$A = [T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \quad ; \quad \beta = \{e_j\}_{j=1}^3$$

Find eigenvalues (& eigenvectors) of A.

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\therefore$ eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

$$\lambda_1 = 1 \to N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = Sp\Big\{\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1}\Big\} \; ; \; T(v_1) = v_1$$

$$\lambda_2 = 2 \to N\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = N\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = Sp\Big\{\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{v_2}\Big\} \; ; \; T(v_2) = 2v_2$$

$$\lambda_3 = 3 \to N\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ 2x_1 \\ x_1 \end{pmatrix} \right\} = Sp\Big\{\underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{v_3}\Big\} \; ; \; T(v_3) = 3v_3$$

$$\therefore \; w \cdot \; [T]_\gamma^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}$$

修改

OR $$\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}$$

and $$\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix}^{n}=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2^n&0\\0&0&3^n\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}.$$ //

Of course, things are not always this nice:

we can't always solve $\overbrace{\det(A-\lambda I)}^{\text{characteristic polynomial}}=0$ on $\mathbb{R}$,

and even if we do, we don't always get n distinct roots.

Moreover, even if λ_j is a root of algebraic multiplicity m_j (that is, $\det(A-\lambda I)=(\lambda-\lambda_j)^{m_j}\cdots$) it doesn't always give m_j linearly independent eigenvectors ...

eg.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\det(A - \lambda I_3) = (1-\lambda)^2(3-\lambda) \qquad \left(\begin{matrix} A-\lambda I \text{ is} \\ \text{upper } \triangle \end{matrix} \right)$$

$$\lambda_1 = 3 \rightsquigarrow A - 3I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{nullity} = 1$$

$$\lambda_2 = 1 \rightsquigarrow A - I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{nullity} = 1$$

$\therefore$ only get 2 eigenvectors, can't form eigenbasis $\Rightarrow$ A not diagonalizable.

eg.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I_4) = \det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ -1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda \det \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} - \det \begin{pmatrix} -1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \lambda^4 + 1 \rightsquigarrow \text{no real solution!}$$

$\therefore$ A is not diagonalizable

Facts:

① If $\det(A - \lambda I_n)$ has n distinct roots $\Rightarrow$ A is diagonalizable (HW. problem)

② If $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is symmetric $(A = A^T)$, then A is diagonalizable

③ If $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is "self-adjoint" $(a_{ji} = \overline{a_{ij}})$, then A is diagonalizable.

Next semester.

– End of Fall 2014 –