(curves)

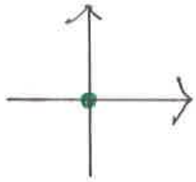
III Conic Sections & Quadric Surfaces

* Conic Sections: Traces in $\mathbb{R}^2$ satisfying equation of the form $P(x,y) = Ax^2+Bx+Dy^2+Ey+F=0$

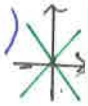
- Prepare Transparency copy of Fig. 11.36 in Thomas p. 658.

Degenerate cases (not really "curves")

$B=E=F=0$ & $AD>0$ (same sign) $\Rightarrow (x,y)=(0,0)$



$B=E=F=0$ & $AD<0$ (opposite sign) $\Rightarrow y=\pm\sqrt{-\,}\,x$ (two lines)



$A=D=0 \Rightarrow Bx+Ey+F=0 \Rightarrow$ one line

Suppose none of is true...

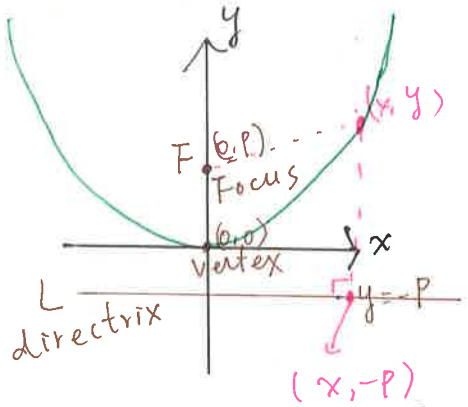
∴ We discuss three cases:

Ⓐ $A=0$ & $D\neq 0$ (OR $A\neq 0$, $D=0$) $\longrightarrow$ parabola

Ⓑ $AD>0$ & both $\neq 0$ & same sign $\to$ ellipse (circle when $a=c$)

Ⓒ $AD<0$ & both $\neq 0$ & opposite sign $\to$ hyperbola

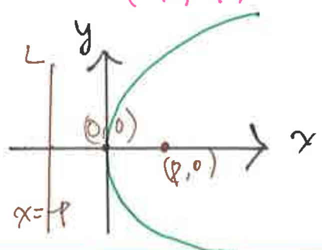
case Ⓐ Parabola: points equidistant to a fixed point & a line



Find all (x,y) equidistant to to L & F, i.e.

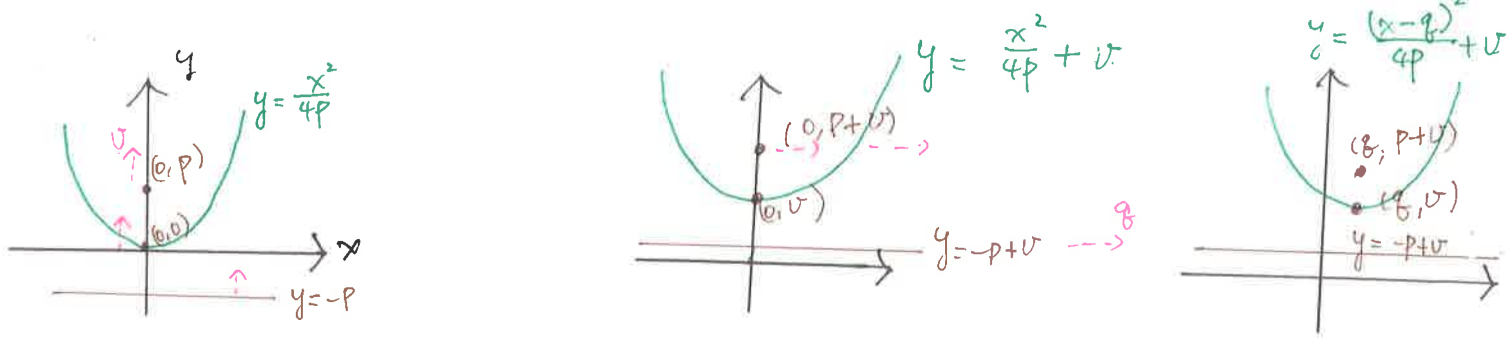
$$x^2+(y-p)^2 = (y+p)^2 \Rightarrow y=\frac{x^2}{4p}. \quad \begin{pmatrix} A\neq 0 \\ D=0 \end{pmatrix}$$

Similarly, for , we get



$$x=\frac{y^2}{4p} \quad \begin{pmatrix} A=0 \\ D\neq 0 \end{pmatrix}$$

b, d, e will appear by "shifting"



$y = \frac{(x-q)^2}{4p} + v$ is the parabola with

- focus $(q, p+v)$
- vertex (q, v)
- directrix $y = -p+v$

Similarly,

$x = \frac{(y-q)^2}{4p} + v$ " " "

- focus $(p+v, q)$
- vertex (q, v)
- directrix $x = -p+v$

eg. $y = 2x^2 + 4x + 7$

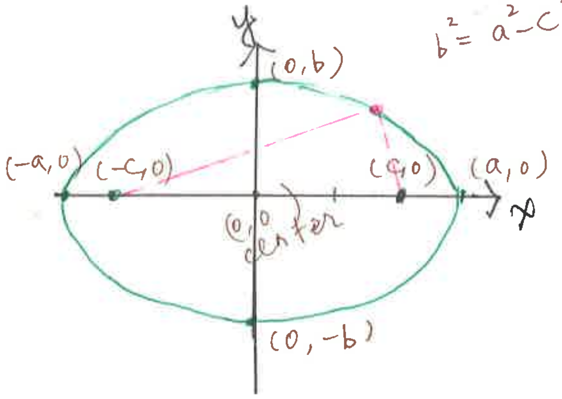
Exercise: write down eq. of parabolas with general focus and directrix.

* Ellipses

set of points in $\mathbb{R}^2$ whose distances to two fixed points (foci) have constant sum.

Simplified case: foci : $(c,0)$ & $(-c,0)$

constant sum : $2a$

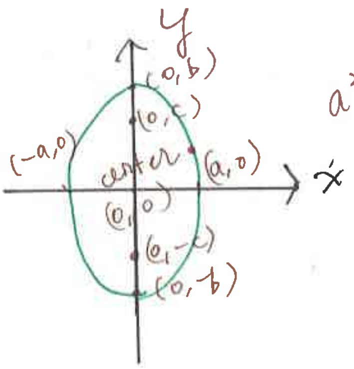


$b^2 = a^2 - c^2$

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{\underbrace{a^2-c^2}_{b^2}} = 1 \qquad (a>b)$$

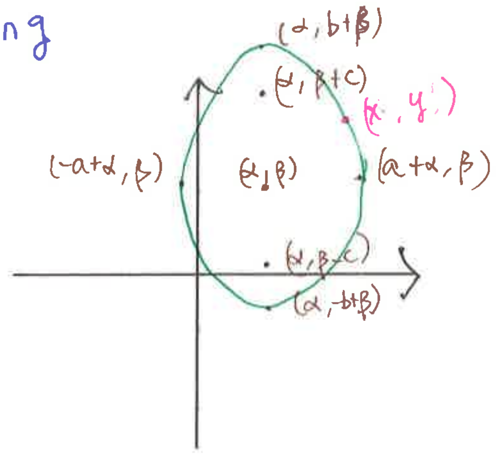
Similarly, when foci are on y-axis.



$a^2 = b^2 - c^2$

$$\frac{x^2}{\underbrace{b^2-c^2}_{a^2}} + \frac{y^2}{b^2} = 1 \qquad (b>a)$$

Shifting



$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$

Conclusion: $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{b^2} = 1$ (*)

is an ellipse, centered at (α, β)

with foci $\begin{cases} (\alpha \pm c, \beta) : c^2 = a^2 - b^2 & \text{if } a>b \\ (\alpha, \beta \pm c) : c^2 = b^2 - a^2 & \text{" } a<b. \end{cases}$

eg III 1 Graph $x^2-4y^2+2x+8y-7=0$

coeff. of x^2 & y^2 opposite signs → hyperbola

$$x^2+2x+1-4(y^2-2y+1) = 7+1-4$$

$$(x+1)^2-4(y-1)^2 = 4 \quad \text{OR}$$

$$\frac{(x+1)^2}{\underbrace{4}_{a^2}} - \frac{(y-1)^2}{\underbrace{1}_{b^2}} = 1 \longrightarrow c^2=5 \text{ OR } c=\sqrt{5}$$

∴ hyperbola centered at $(-1,1)$

foci $(-1\pm\sqrt{5},\ 1)$

eg III 2. Graph $2x^2-6x+6y^2+2y+\frac{3}{4}=0$

coefficients of x^2 & y^2 same sign ⇒ ellipse

$$2\left[x^2-3x+\left(\tfrac{3}{2}\right)^2\right]+6\left[y^2+\tfrac{y}{3}+\tfrac{1}{36}\right] = -\cancel{\tfrac{3}{4}}+\tfrac{9}{2}+\tfrac{1}{6} = \frac{\cancel{52}\,4}{6} = 1\tfrac{\cancel{6}}{3}$$

$$2\left(x-\tfrac{3}{2}\right)^2+6\left(y+\tfrac{1}{6}\right)^2 = \cancel{\tfrac{26}{3}}\ 4$$

OR

$$\frac{\left(x-\frac{3}{2}\right)^2}{\underbrace{2\ \cancel{\tfrac{13}{52}}}_{a^2}} + \frac{\left(y+\frac{1}{6}\right)^2}{\underbrace{\tfrac{2}{3}\ \cancel{\tfrac{11}{52}}}_{b^2}} = 1$$

$$c^2 = a^2-b^2 = \cancel{\tfrac{1}{52}} \Rightarrow c = \sqrt{\cancel{\tfrac{1}{52}}}$$

∴ ellipse with center $\left(\frac{3}{2}, -\frac{1}{6}\right)$

and foci $\left(\frac{3}{2}\pm\sqrt{\frac{1}{52}},\ -\frac{1}{6}\right)$ //

* Conics in Polar Coordinates

Eccentricity - a measure of how a conic section
e differs from a circle (i.e. $e=0$ for circle)

Many versions for e, we consider a unified set of ingredients: focus & directrix D.

claim: A conic section is a set of points P,
s.t. $\overline{PF} = e \cdot \overline{PD}$.

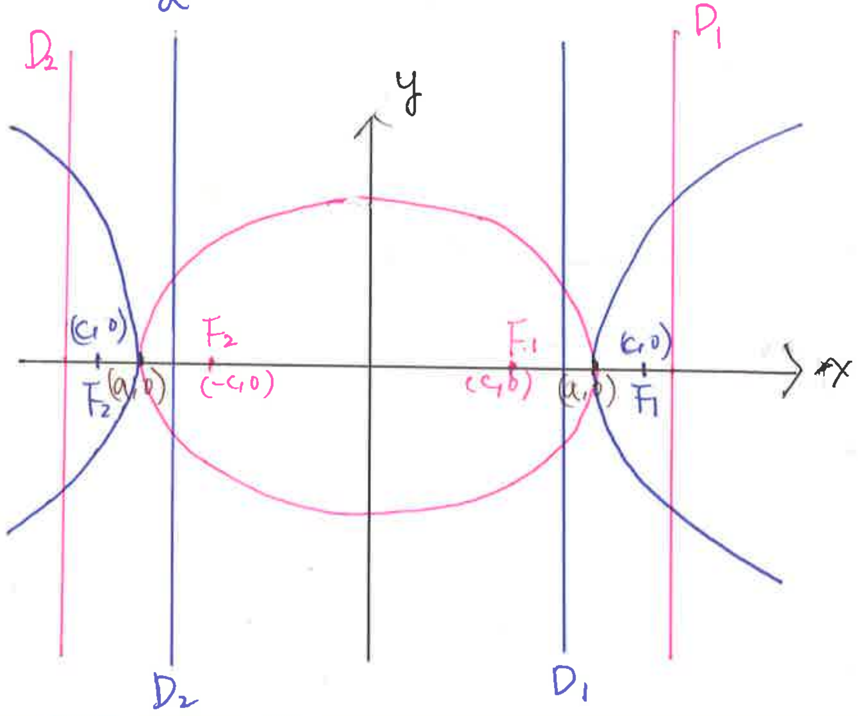
$e=0$: circle
$e \in (0,1)$: ellipse
$e=1$ $\begin{cases} \text{line segment} ; \text{ if } F \in D. \\ \boxed{\text{parabola} ; \text{ if } F \notin D} \end{cases}$ have seen before
$e \in (1,\infty)$: hyperbola
$e=\infty$: line (directrix itself)

consider ellipse & hyperbola w/ equations $(a>b)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \& \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$(c^2 = a^2 - b^2)$ $\quad (c^2 = a^2 + b^2) \Rightarrow F = (\pm c, 0)$

Let $e = \frac{c}{a}$ in both cases & consider $D: x = \pm \frac{a}{e}$



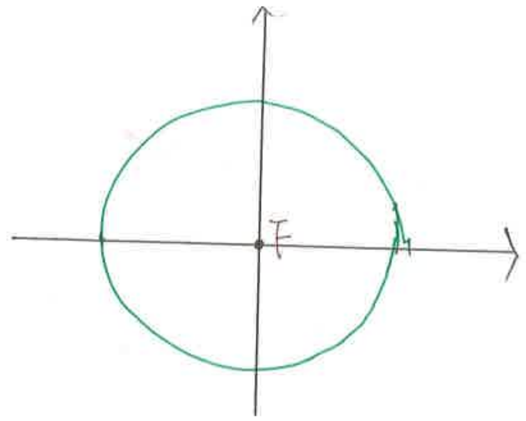
Discuss:

- $c=0$ (circle) ($e=0$)
- $c \in (0,a)$

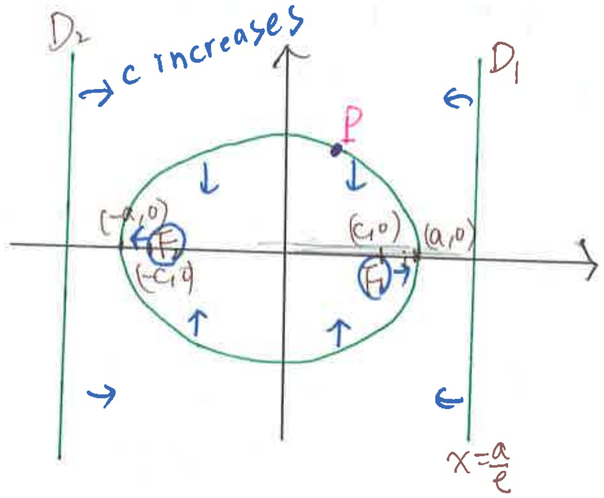
  P on ellipse
  $\Updownarrow$
  $PF_i = e\, PD$

- $c=a$
- $c \in (a,\infty)$
  PF_i

$c=0$; $D: x = \pm\infty$
$(a=b)$
$e=0$



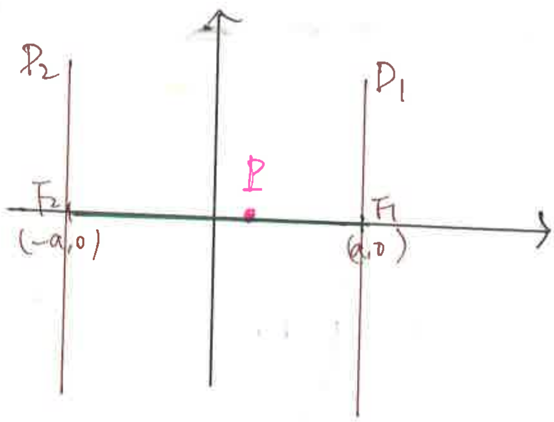
$e \in (0,1)$
$c \in (0,a)$; $D: x = \pm \frac{a}{e}$ $\left(\frac{a}{e} > a\right)$



can show, for any P on ellipse.

$\overline{PF_i} = e \cdot \overline{PD_i}$ $(i=1,2)$

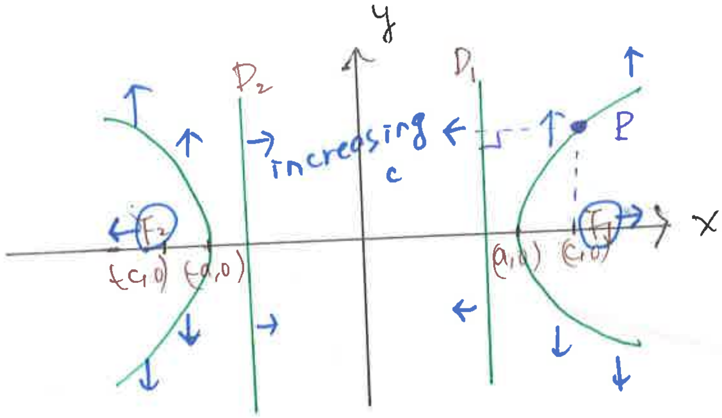
$c=a$, $e=1$, $D: x=\pm a$



clearly

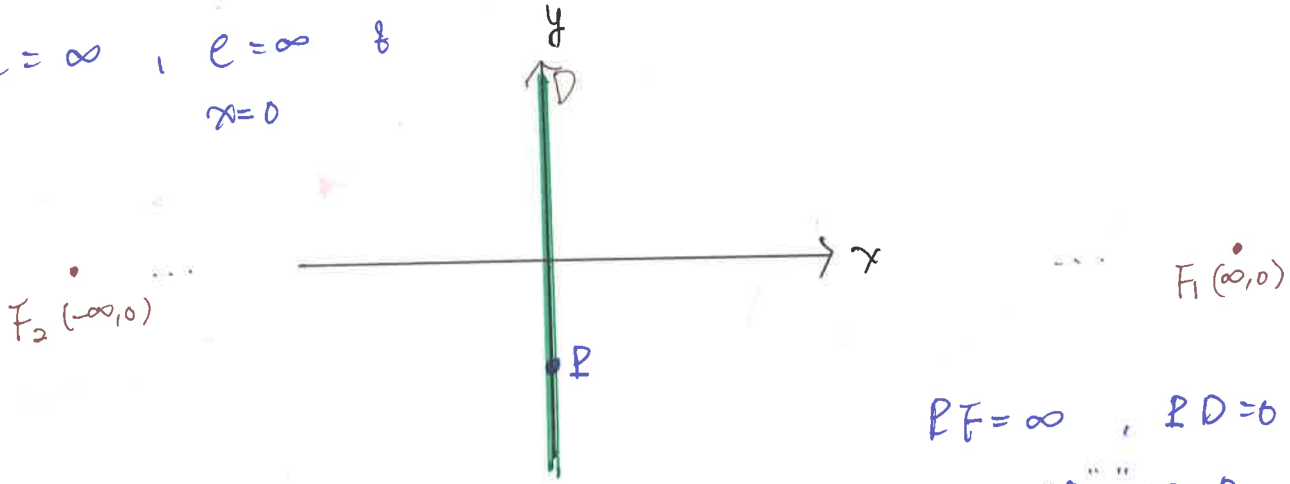
$\overline{PF_i} = \overline{PD_i}$; $i=1,2$

$c \in (a, \infty)$, $e > 1$, $x = \pm \frac{a}{e}$ & $\frac{a}{e} < a$



Again, $PF_i = e\,PD_i$
$\forall P$ on hyperbola

$c = \infty$, $e = \infty$ &
$x = 0$



$PF = \infty$, $PD = 0$
$0 = " \infty \cdot 0 "$.

In polar form, place F_1 (or F_2) at origin

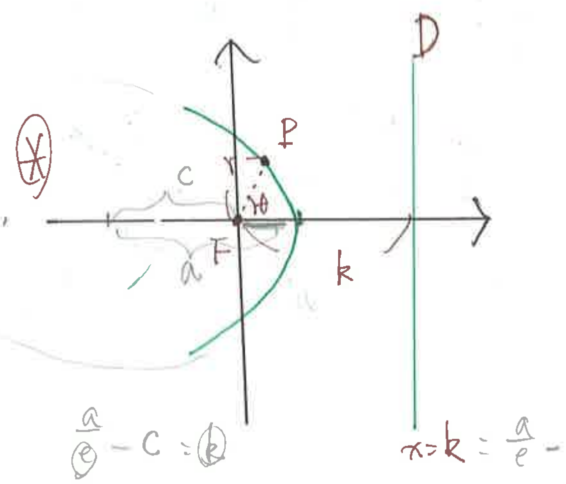
$\Rightarrow$ $PF_1 = r$.

consider vertical directrix D : $x = k > 0$

$\frac{ke}{1+e} = a - c$
$= a - ae$
$= a(1-e)$

$e = \frac{c}{a}$

$c + k = \frac{a}{e}$



$PD = k - r\cos\theta$

$PF = e\,PD$

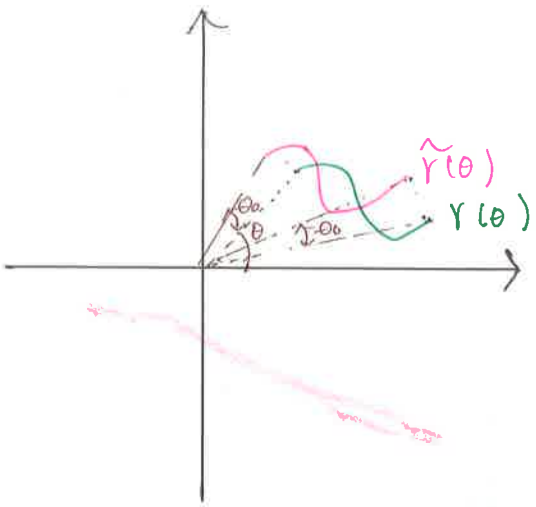
$\Rightarrow$ $r = ek - er\cos\theta$

OR $r = \frac{ke}{1 + e\cos\theta}$

Check: if we choose the other F, D on the left we get $r = \frac{ke}{1 - e\cos\theta}$

Polar equations are also convenient for describing rotation:

Observe: $\tilde{r}(\theta) = r(\theta - \theta_0)$ is the curve formed by rotating $r(\theta)$ by θ_0, counterclockwise. (c.c.w.)



In particular, horizontal directrix $y=k$ & focus $(0,0)$ is obtained from rotating ⊛ by θ_0, c.c.w.

$\therefore$ we get equation

$$r(\theta) = \frac{ke}{1+e\cos(\theta-\frac{\pi}{2})} = \frac{ke}{1+e\sin\theta}$$

(& similarly for $y=-k$, $F=(0,0)$,

$$r(\theta) = \frac{ke}{1-e\sin\theta}$$

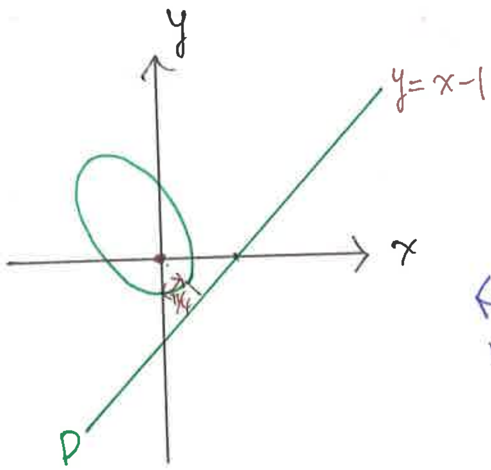
)

eg// $r = \dfrac{25}{10 + 20\cos\theta}$ determine type of conic, and directrix (w/ focus $= (0,0)$)

$$r = \frac{25/10}{1 + 2\cos\theta} = \frac{\overset{e}{2} \cdot \overset{k}{\frac{25}{20}}}{1 + \underset{e}{2} \cos(\theta + 0)}$$

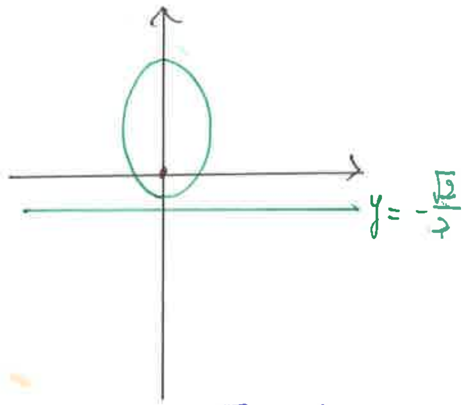
vertical directix
$x = +k = \frac{25}{20}$.

eg// write down polar equation of the conic w/ focus $(0,0)$ & directix $y = x - 1$, $e = \frac{1}{2}$



rotation by $\theta_0 = \frac{\pi}{4}$

consider



$$\tilde{r} = \frac{\frac{\sqrt{2}}{2} \cdot \frac{1}{2}}{1 - \frac{1}{2}\sin\theta} = \frac{\sqrt{2}}{2 - \sin\theta}$$

$$r(\theta) = \tilde{r}(\theta - \tfrac{\pi}{4})$$
$$= \frac{\sqrt{2}}{2 - \sin(\theta - \frac{\pi}{4})} \quad //$$

* Quadric Surfaces

Traces in $\mathbb{R}^3$ of the quadratic equation

$$Ax^2+By^2+Cz^2+Dx+Ey+Fz+G=0$$

(omit mixed terms now)

and $(A,B,C) \neq (0,0,0)$

Let's assume $D=E=F=0$ (if not, complete the square and shift the plot with $D=E=F=0$),

Generally not easy; the most common strategy is to fix one variable and see what kind of conic sections we get.

A few obvious ones:

Type A: $A, B, C \neq 0$

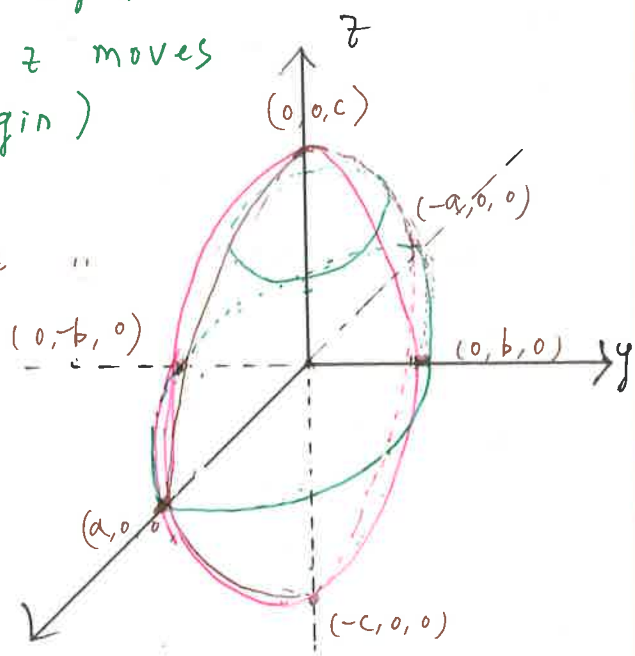
A1. Ellipsoid $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

$$\left(\Rightarrow \begin{array}{l} -a \le x \le a \\ -b \le y \le b \\ -c \le z \le c \end{array} \right)$$

fix $z \Rightarrow$ ellipse parallel to xy plane
(shrinking when z moves away from origin)

fix $y \Rightarrow$ " " " xz "
(" ......)

fix $x \Rightarrow$ " " " yz



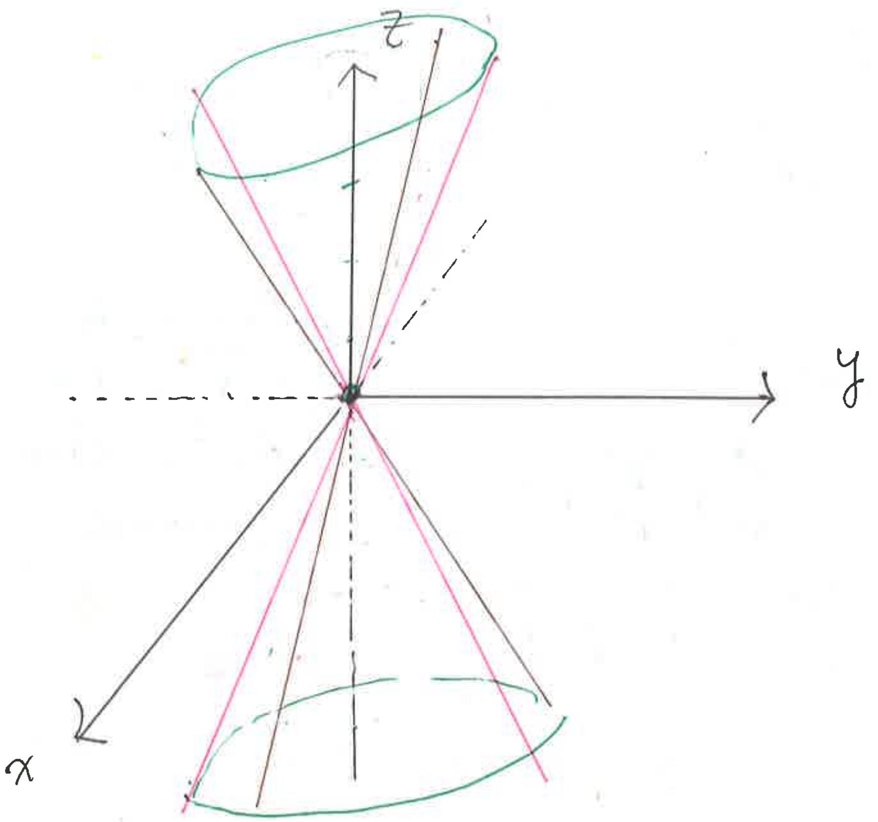
Q11 what about $a=b=c$?

A2. Elliptical cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

fix z $\Rightarrow$ ellipse // xy plane
shrinking as $z \to 0$ (& $z=0 \Rightarrow (x,y) = (0,0)$)

fix $y=0 \Rightarrow$ 2 lines // xz plane ($z = \pm \frac{c}{a} x$)
fix $x=0 \Rightarrow$ " " " yz " ($z = \pm \frac{c}{b} y$)



What about $y = y_0 \neq 0$?
$x = x_0 \neq 0$?

A3. Hyperboloic of two sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \left(\frac{z^2}{c^2} \geq 1 \Rightarrow \begin{array}{l} z \geq c \\ z \leq -c \end{array} \text{ ie } z \notin (-c, c) \right)$$

fix $z = z_0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z_0^2}{c^2}-1$

ellipse // xy plane (larger as z moves away from 0)

fix $y = y_0 \Rightarrow \frac{z^2}{c^2}-\frac{x^2}{a^2}=1+\frac{y_0^2}{b^2}$

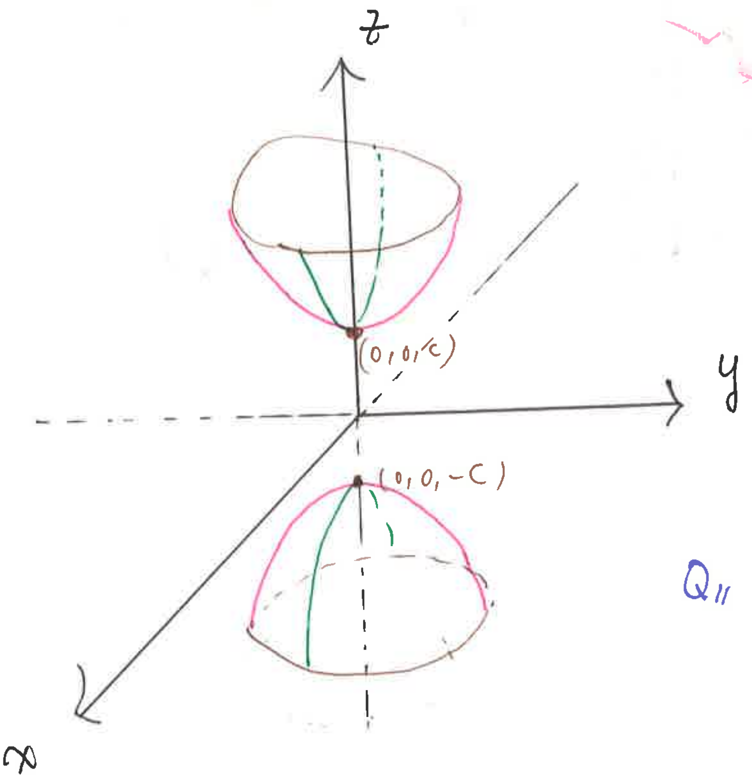
hyperbolic // xz plane

(further apart as y moves away from 0)

fix $x = x_0 \Rightarrow \frac{z^2}{c^2}-\frac{y^2}{b^2}=1+\frac{x_0^2}{a^2}$

// yz ''

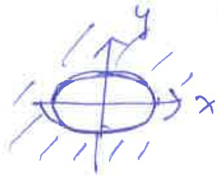
('' '' '' x '')



Q// what about $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$?

Hyperboloid of one sheet

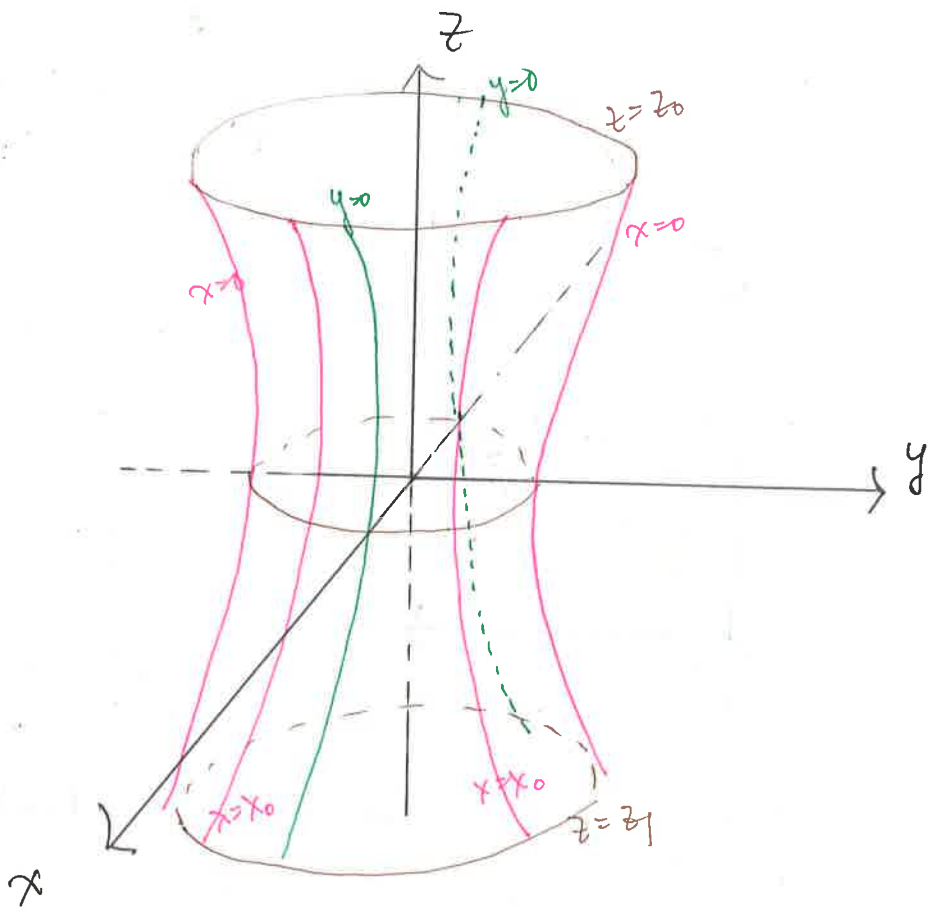
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \left(\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}\geq 1 \rightsquigarrow\right.$$



fix $z=z_0$ $\Rightarrow$ ellipse // xy plane
(larger as z moves away from 0)

fix $y=y_0$ $\Rightarrow$ hyperbola // xz plane
(further apart as y moves away from 0)

fix $x=x_0$ $\Rightarrow$ " " yz "
(" " " x
" " " ")



Type B: one of A, B, C = 0

B1. Elliptical Paraboloid.

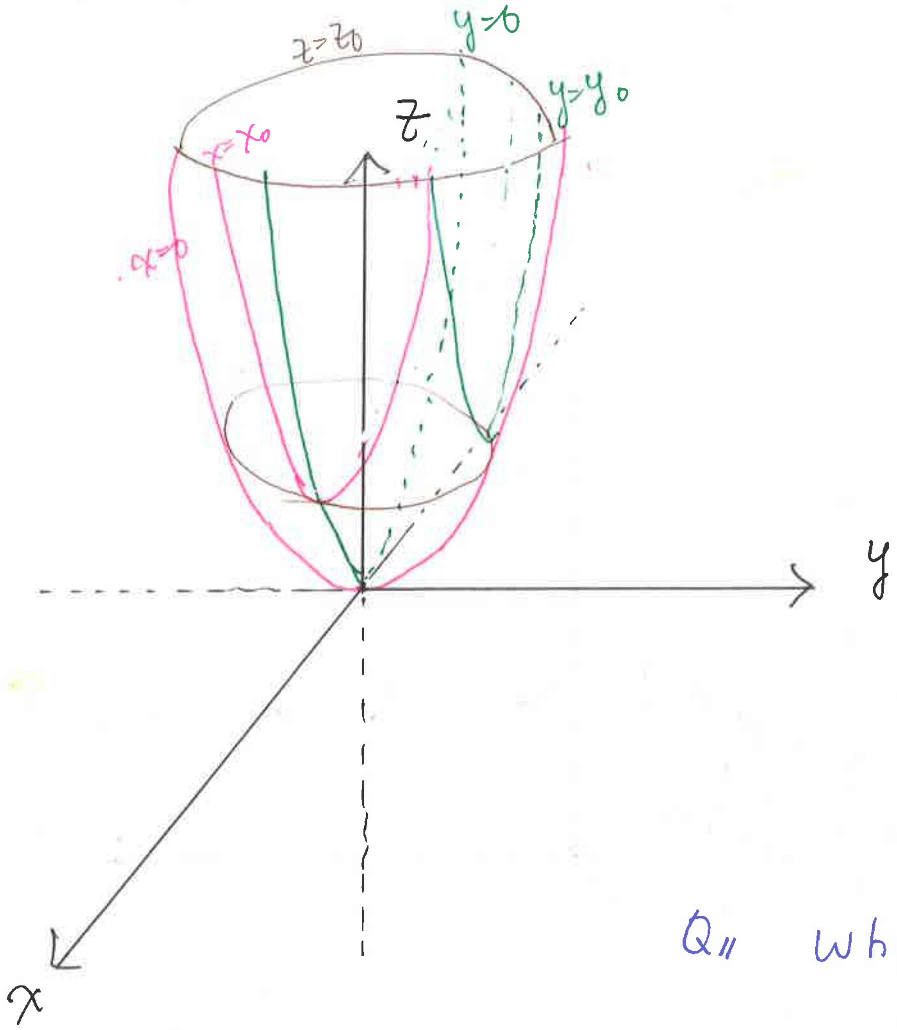
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \qquad (\Rightarrow z \geq 0)$$

assume $c > 0$

fix $z = z_0$, ellipse // xy plane

fix $y = y_0$, parabola // xz plane $\quad z = \frac{c}{a^2}x^2$

fix $x = x_0$, " " // yz plane $\quad z = \frac{c}{b^2}y^2$



Q. What about $c < 0$?

B2.

Hyperbolic paraboloid.

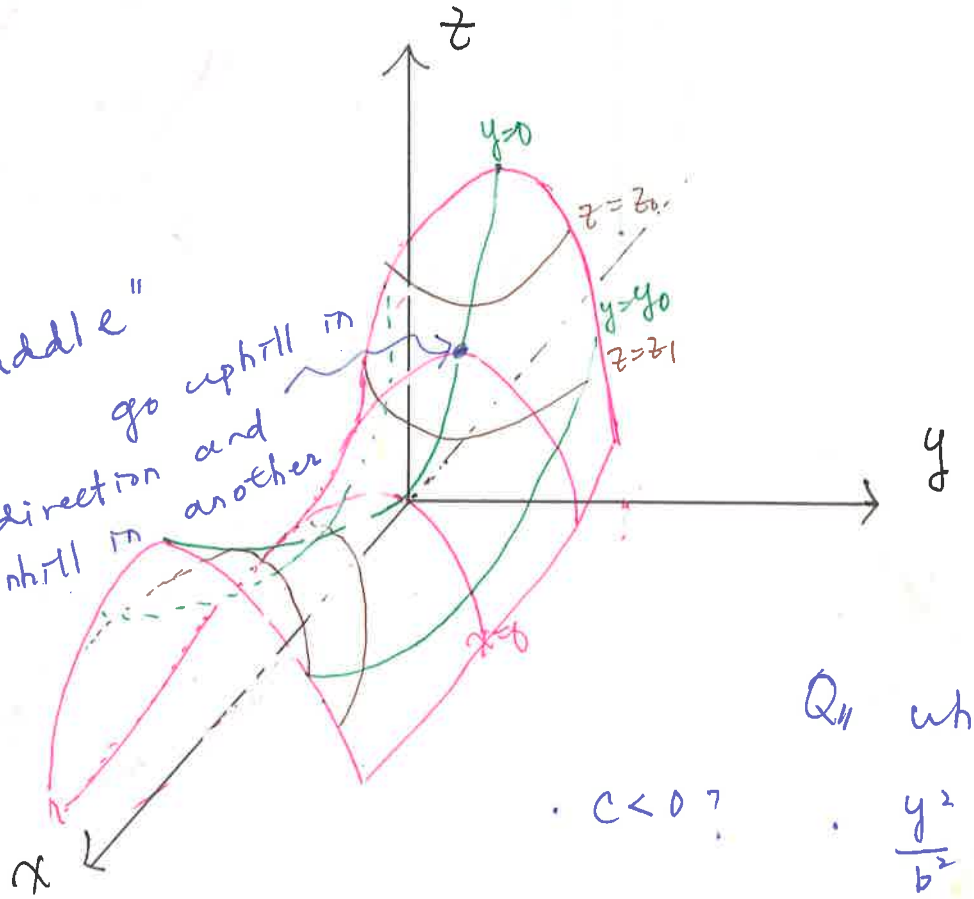
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

assume $c > 0$.

fix $z = z_0 \neq 0$, hyperbola // xy plane — open toward x axis
(further apart as z moves away from 0)

fix $y = y_0$, parabola $z = \frac{c}{a^2}x^2 - \frac{c}{b^2}y_0^2$ // xz plane

fix $x = x_0$ " $z = -\frac{c}{b^2}y^2 + \frac{c}{a^2}x_0^2$ // yz plane



" saddle "

go uphill in one direction and downhill in another

Q// what about

- $c < 0$?
- $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$?
- $\frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{x}{a}$?

Type C : only one square term

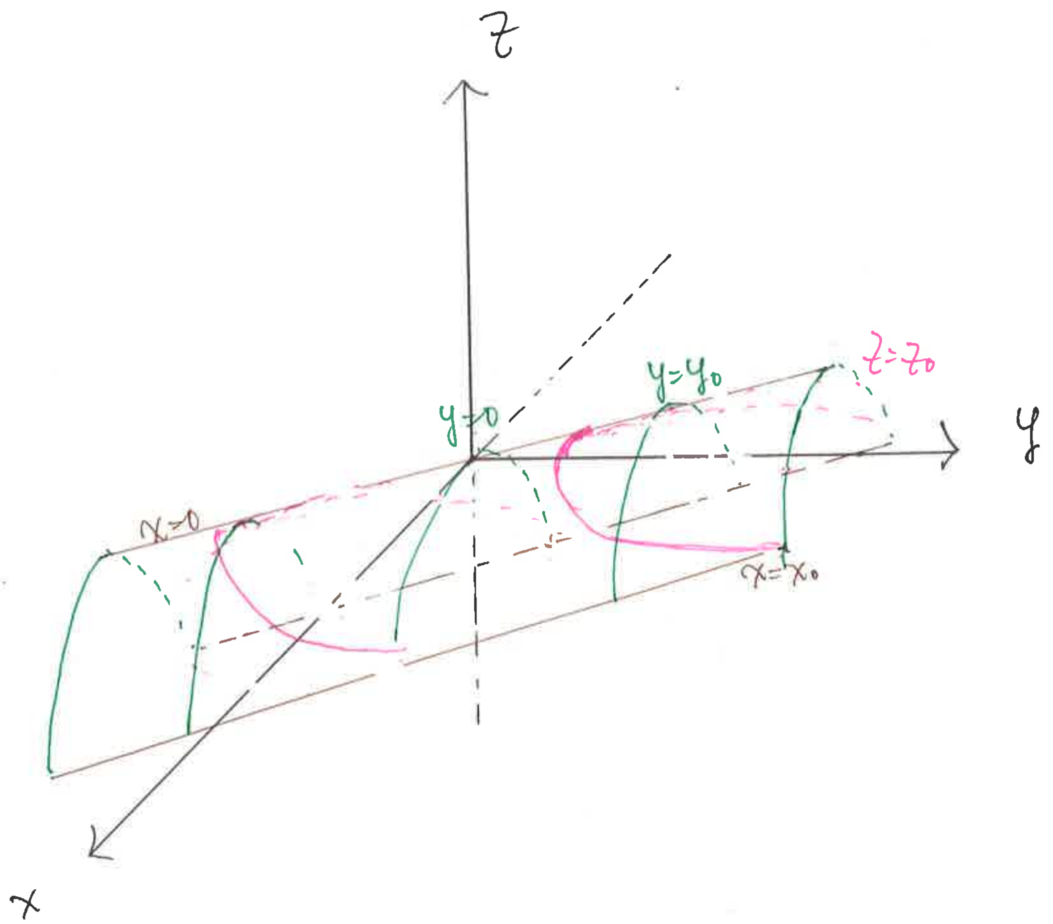
C1. $\frac{x^2}{a^2} - \frac{y}{b} + \frac{z}{c} = 0$ assume $b, c > 0$

fix $x = x_0$ $\Rightarrow$ line // yz plane

$z = \frac{c}{b} y - \frac{c}{a^2} x_0^2$

fix $y = y_0$ $\Rightarrow$ parabola $z = -\frac{c}{a^2} x^2 + \frac{c}{b} y_0$ // xz plane

fix $z = z_0$ $\Rightarrow$ " $y = \frac{b}{a^2} x^2 + \frac{z_0}{c}$ // xy plane

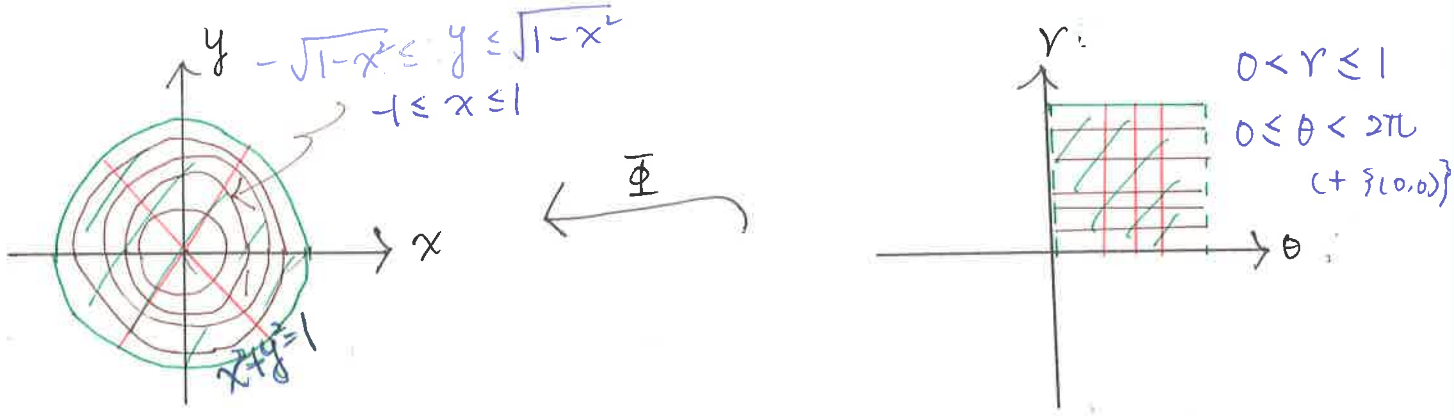


online 3-D Function Grapher:

www.livephysics.com/tools/mathematical-tools/online-3-d-function-grapher.

* More Change of Coordinates & Descriptions of Domains

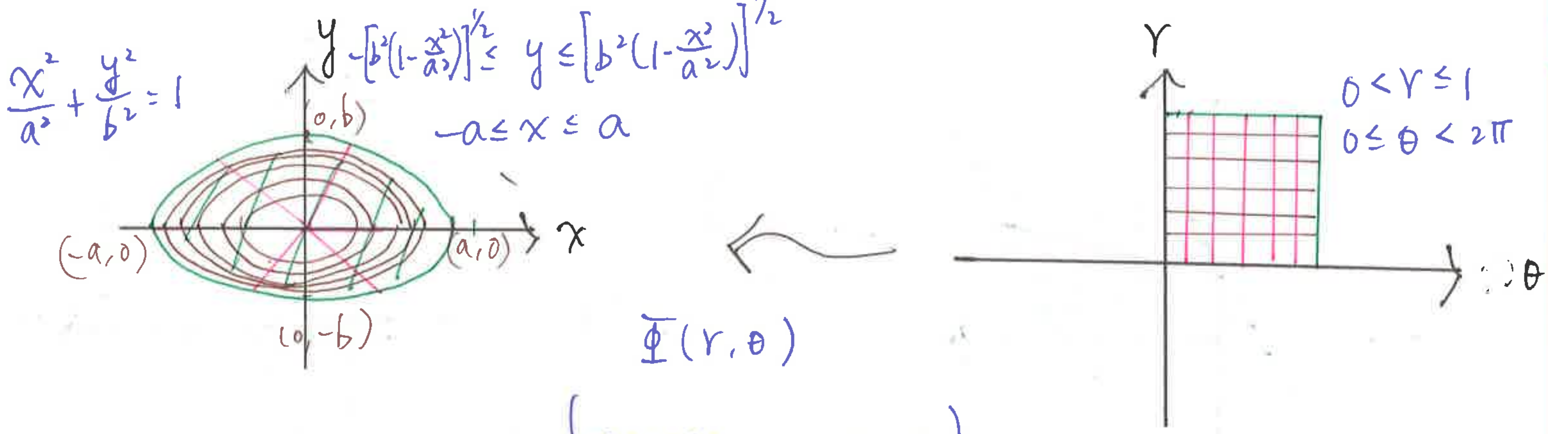
Recall polar coordinates:



Forget about boundary in $r\theta$ plane. we develope some more changes of coordinate similar to what we have seen:

consider disc / ball - like domains:

1. Ellipse.

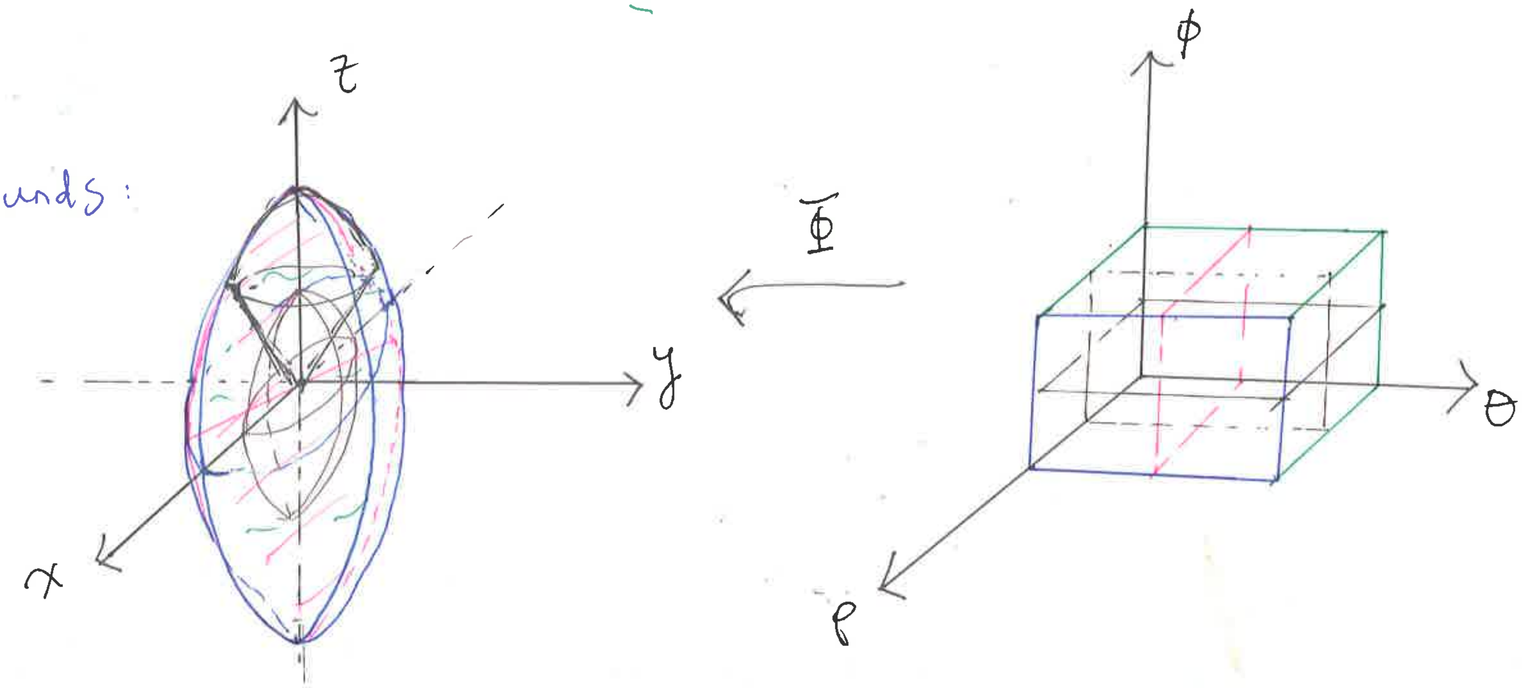


$$\Phi(r,\theta) = (ar\cos\theta,\ br\sin\theta)$$

Similarly for 3D plot:

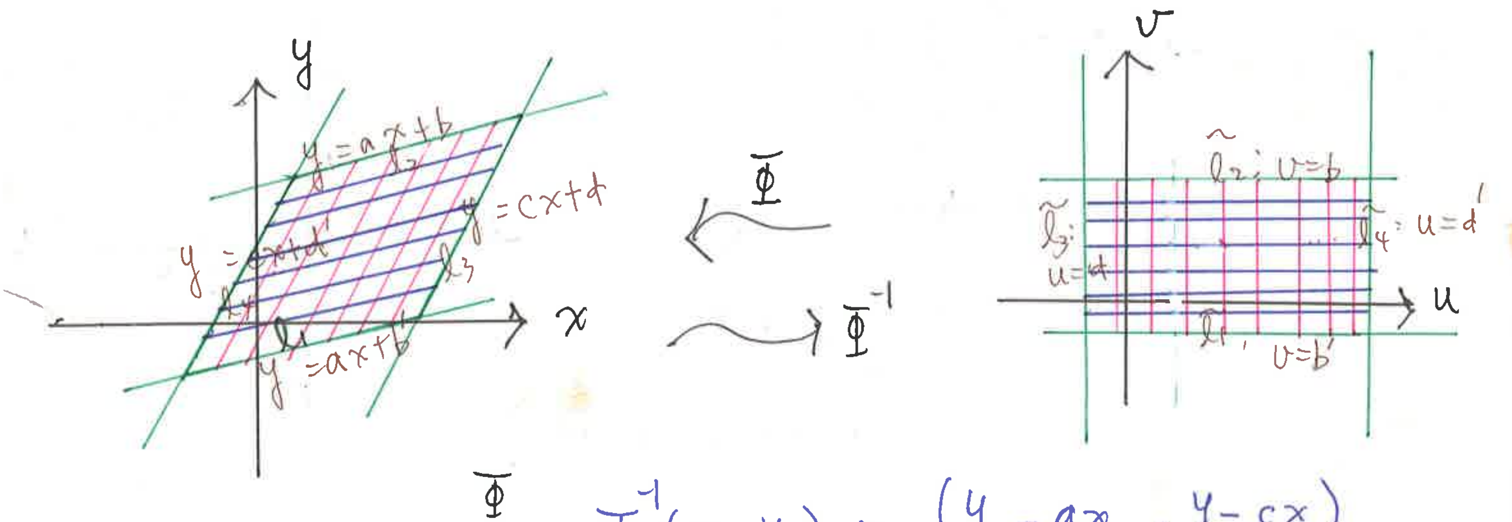
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\underset{\leq}{=}1$$

Bounds:



$$\Phi(\rho,\theta,\phi) = (a\rho\sin\phi\cos\theta,\ b\rho\sin\phi\cos\theta,\ c\rho\cos\phi)$$

Parallelogram.



$$\Phi^{-1}(x,y) = (\underbrace{y-ax}_{u(x,y)},\ \underbrace{y-cx}_{v(x,y)})$$

$$\begin{cases} v(x,y) = y-ax \\ u(x,y) = y-cx \end{cases}$$

$\Rightarrow\ \Phi^{-1}(\overset{l_2}{l_1}) = \{(\overset{b}{b'}, \sim)\}$; $\Phi^{-1}(l_3, l_4) = \{(\sim, \overset{d'}{d})\}$;

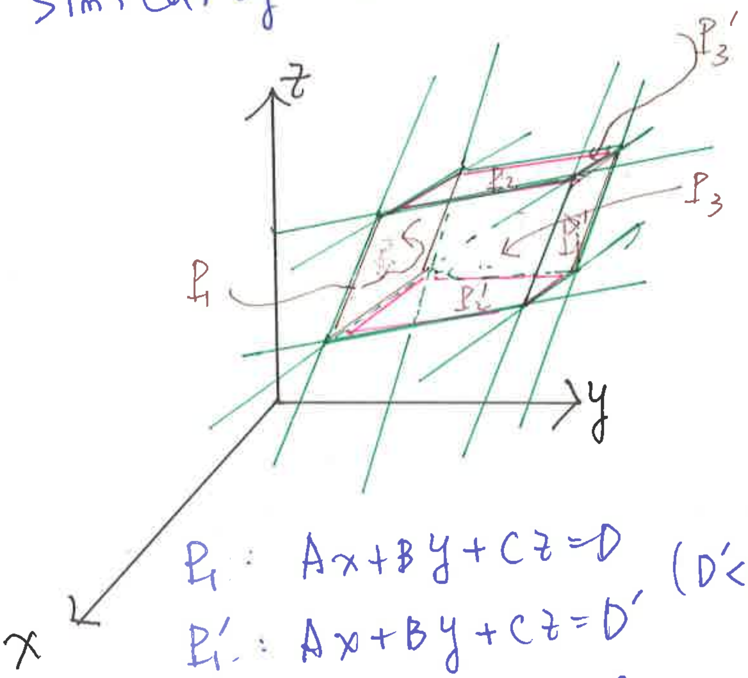
$$\Phi^{-1}(x,y) = (\underbrace{y-cx}_{u}, \underbrace{y-ax}_{v})$$

$$\begin{cases} u = y - cx \\ v = y - ax \end{cases} \rightarrow \underbrace{\begin{cases} x = \frac{1}{a-c}(u-v) \\ y = \frac{1}{a-c}(au-cv) \end{cases}}_{\Phi(u,v)}$$

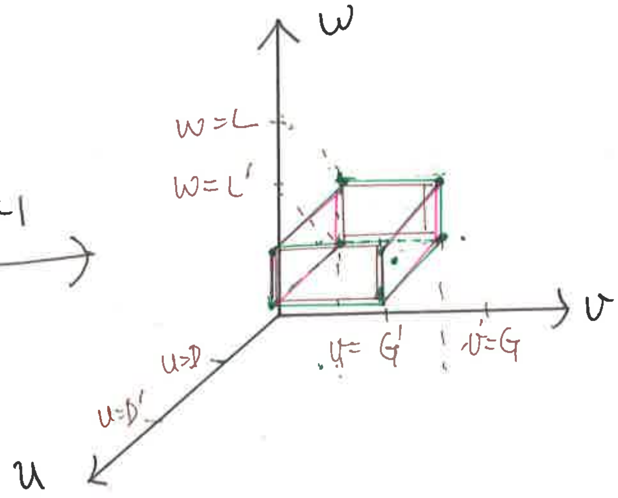
$$\Phi^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -c & 1 \\ -a & 1 \end{pmatrix}}_{A_\Phi}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y-cx \\ y-ax \end{pmatrix} \begin{matrix} \to u \\ \to v \end{matrix}$$

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{a-c} & \frac{-1}{a-c} \\ \frac{a}{a-c} & \frac{-c}{a-c} \end{pmatrix}}_{A_\Phi^{-1}}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{u}{a-c} - \frac{v}{a-c} \\ \frac{au}{a-c} - \frac{cv}{a-c} \end{pmatrix}$$

Similarly in 3D:



$\xrightarrow{\Phi^{-1}}$



P_1 : $Ax + By + Cz = D$ $(D' < D)$

P_1' : $Ax + By + Cz = D'$

P_2 : $Dx + Ey + Fz = G$ $(G' < G)$

P_2' : $Dx + Ey + Fz = G'$

P_3 : $Hx + Iy + Kz = L$ $(L' < L)$

P_3' : $Hx + Iy + Kz = L'$

$$\Phi^{-1}(x, y, z) = (Ax+By+Cz,\ Dx+Ey+Fz,\ Hx+Iy+Kz)$$

$$\Phi^{-1}(P_1) = \{(D, \cdots, \cdots)\} \qquad \Phi^{-1}(P_1') = \{(D', \cdots, \cdots)\}$$

$$\Phi^{-1}(P_2) = \{(\cdots, G, \cdots)\} \qquad \Phi^{-1}(P_2') = \{(\cdots, G', \cdots)\}$$

$$\Phi^{-1}(P_3) = \{(\cdots, \cdots, L)\} \qquad \Phi^{-1}(P_3') = \{(\cdots, \cdots, L')\}$$

$$\Phi^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} A & B & C \\ D & E & F \\ H & I & K \end{pmatrix}}^{A_{\Phi^{-1}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and $$\Phi\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \\ H & I & K \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

~~s/o~~ ie solve

$$\begin{cases} u = Ax + By + Cz \\ v = Dx + Ey + Fz \\ w = Hx + Iy + Kz \end{cases}$$ to get

$$\begin{aligned} x &= \_\_\, u + \_\_\, v + \_\_\, w \\ y &= \_\_\, u + \_\_\, v + \_\_\, w \\ z &= \_\_\, u + \_\_\, v + \_\_\, w \end{aligned}$$

$$\begin{pmatrix} \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = (A_{\Phi^{-1}})^{-1} = A_{\Phi}$$