II. Coordinates

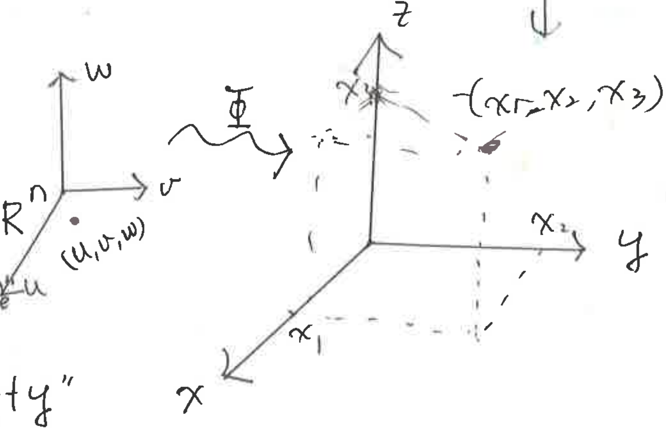
points/vectors

$\mathbb{R}^n := \{\vec{x} = \underbrace{(x_1, \dots, x_n)}_{\text{coordinates of } \vec{x}} \mid x_i \in \mathbb{R} \text{ for all } i\}$ Euclidean n-space

Coordinate system: a geometric way to uniquely label points on rectangular (Cartesian) coordinate system

$\Phi : U \subset \mathbb{R}^n \to \mathbb{R}^n$

- Φ is "one-to-one"
- some other "compatibility" conditions.



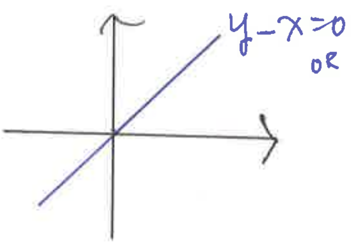
"Right-Hand" coordinate

* Traces (Plots) by equations in $\mathbb{R}^n$:

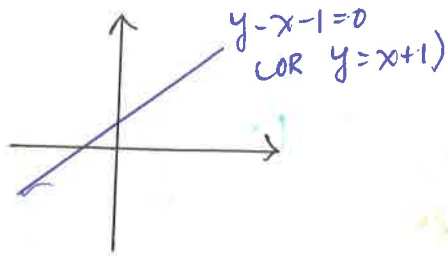
$\{\vec{x} \in \mathbb{R}^n \mid P_1(\vec{x}) = \dots = P_n(\vec{x}) = 0$ & possibly more conditions$\}$ defining equation(s)

usually in $\mathbb{R}^2$, $\vec{x} \to (x, y)$
$\mathbb{R}^3$, $\dots (x, y, z)$ } most of the focus in this course

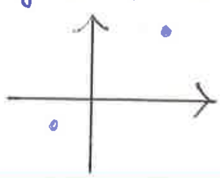
eg II 1: in $\mathbb{R}^2$, $P(x,y) = y - x$,



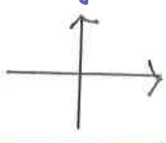
II 2 $P(x,y) = y - x - 1$



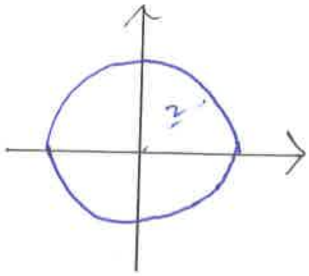
II 3 $P_1(x,y) = y - x$, $P_2(x,y) = x - y^2 + 2$



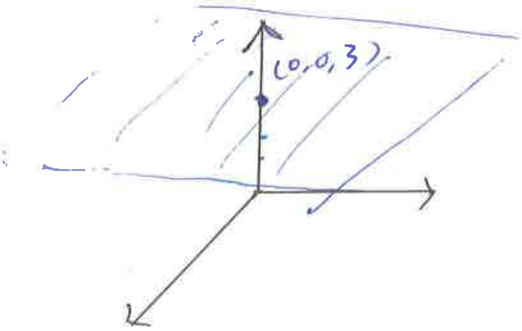
II 4. $P_1(x,y) = y - x$
$P_2(x,y) = y - x - 1$



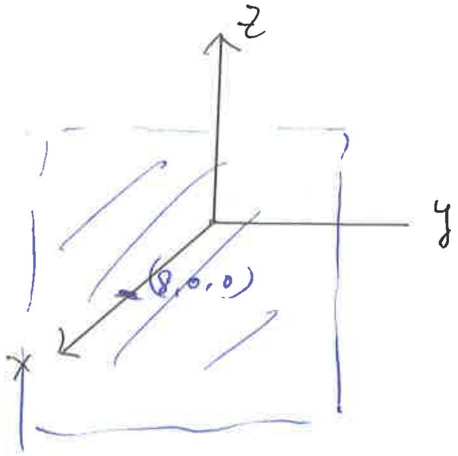
eg. II 5 $\quad P(x,y) = x^2+y^2-4$



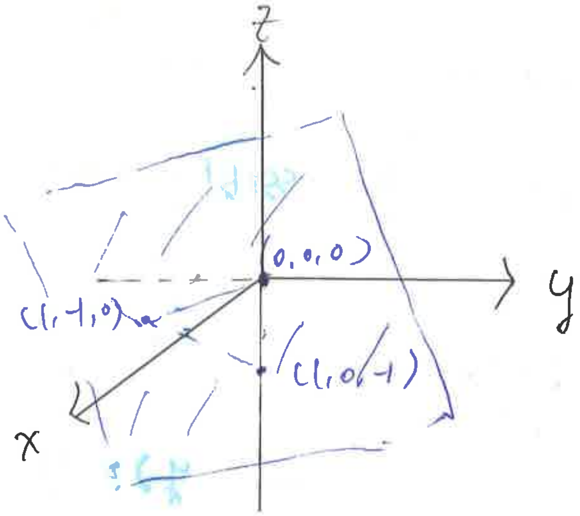
II 6. $\quad P(x,y,z) = z-3$



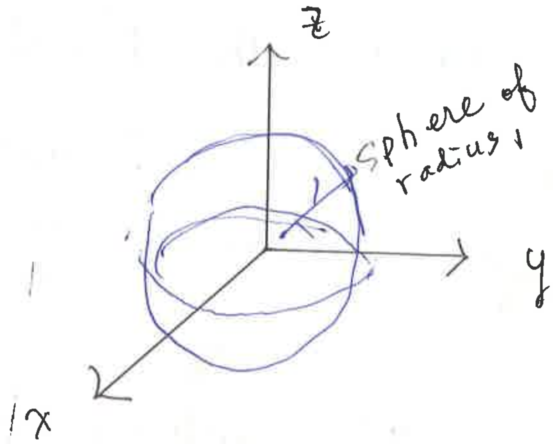
II 7. $\quad P(x,y,z) = x-8$



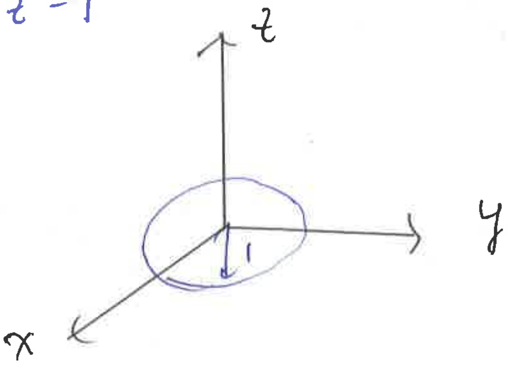
II 8. $\quad P(x,y,z) = x+y+z$



II 9. $\quad P_1(x,y,z)$
$= x^2+y^2+z^2-1$



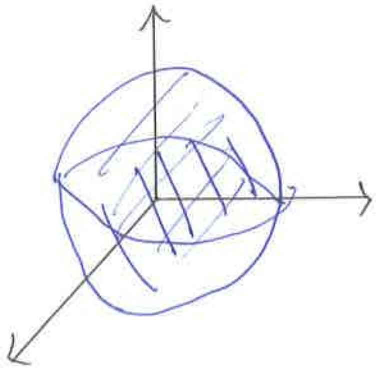
II 10. $\quad P_1(x,y,z) = x^2+y^2+z^2-1$
$P_2(x,y,z) = z$



* Traces of Inequalities

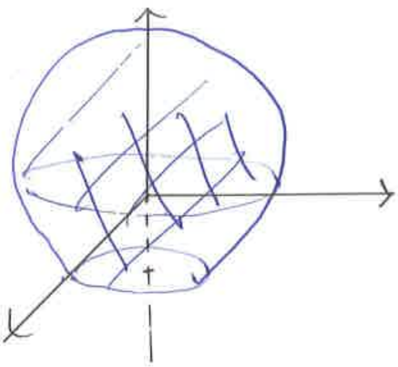
$$\left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; P_i(x) \overset{(>)\;(<)}{\geq,\ \leq} 0 \right\}$$

II 11: $P(x,y,z) = x^2+y^2+z^2-1 \leq 0$



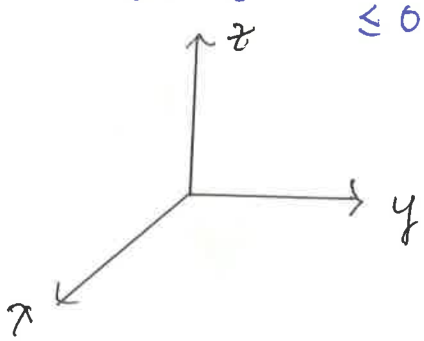
II 12. $P_1(x,y,z) = x^2+y^2+z^2-1 \leq 0$

$P_2(x,y,z) = z+1 \geq 0$



II 15. $P_1(x,y,z) = x \geq 0$

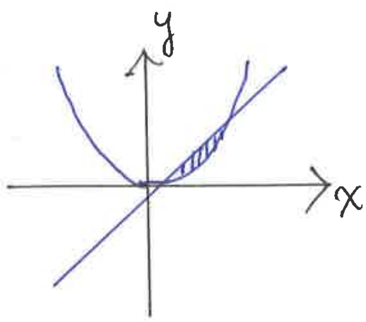
$P_2(x,y,z) = x+1 \leq 0$



II 13. (in $\mathbb{R}^2$)

$P_1(x,y) = y - x^2 \geq 0$

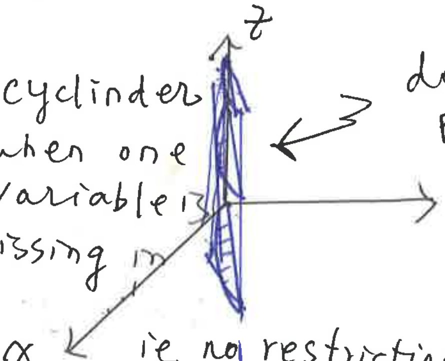
$P_2(x,y) = y - x \leq 0$



II 14 (in $\mathbb{R}^3$)

$P_1(x,y,z) = y - x^2 \geq 0$

$P_2(x,y,z) = y - x \geq 0$



cylinder (when one variable is missing in ie no restriction on that ...)

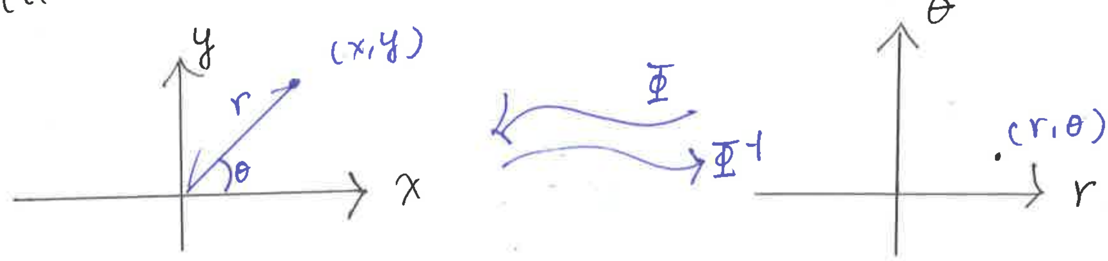
do a Better Job in class

Geometric Observation

- Each "new" equation "reduces" the dimension by at least one., while inequality in general doesn't reduce dimensions (unless they contradict each other, as in eg II 15)

* Points in Different Coordinate System

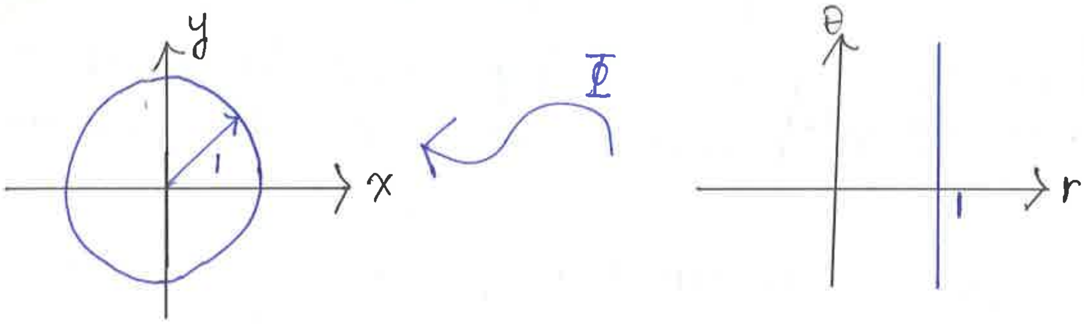
Polar Coordinate System in $\mathbb{R}^2$.



$$\Phi(r,\theta) = (r\cos\theta,\ r\sin\theta)$$
$$\Phi^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ \tan^{-1}\frac{y}{x}\right)$$

polar coordinate systems are suitable to described curves highly dependent on angle and radius (distance to origin).

eg II 1:



Algebraically

$x^2+y^2=1$ ⊛

in r,θ plane,

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 = 1$

⊛ $\boxed{\rightsquigarrow r=1}$

Problems with ⊛:

① what about $r=-1$?

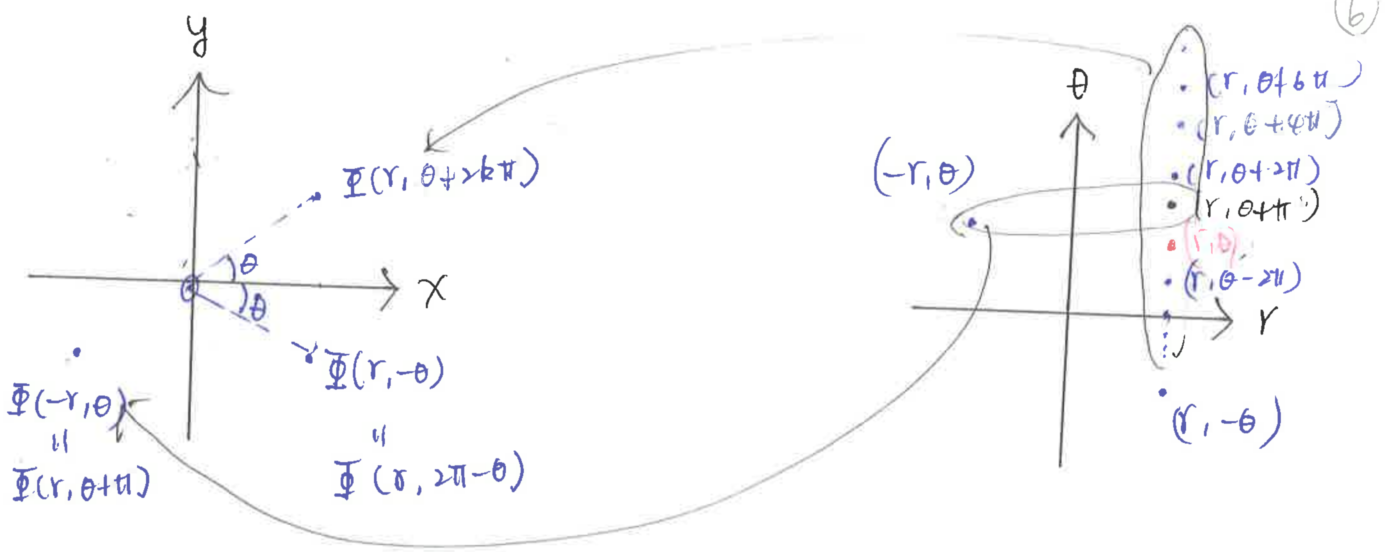
② Φ is not 1-1 !! (ie the "labling" by Φ is not unique)

Observe: $\{ r>0$

$\Phi(-r,\theta) = (-r\cos\theta, -r\sin\theta) = \Phi(r, \theta \pm \pi)$ ($+$: $\theta<\pi$, $-$: $\theta\geq\pi$)

$\Phi(r,-\theta) = (r\cos(-\theta), r\sin(-\theta)) = \Phi(r, 2\pi-\theta)$

$\Phi(r, \theta+2k\pi) = \Phi(r,\theta)$



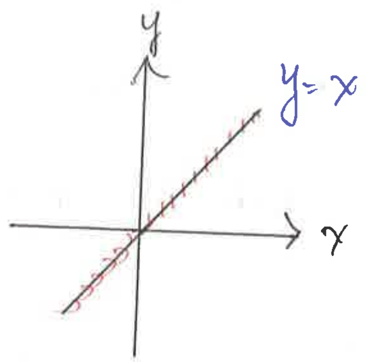
∴ To satisfy one-to-one condition of Φ, we restrict our domain to be $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, when performing change of coordinates.

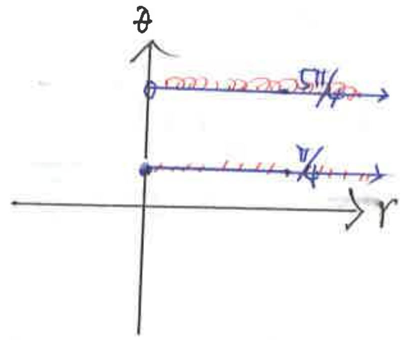
[However, when solving equations of r and θ, we do not restrict any possible value]

Special care: origin $(0,0)$ on x-y plane.

always exclude "$r=0$" when doing change of coordinate, since $\Phi(0,\theta) = (0,0)$ for all θ and we don't have 1-1.

eg

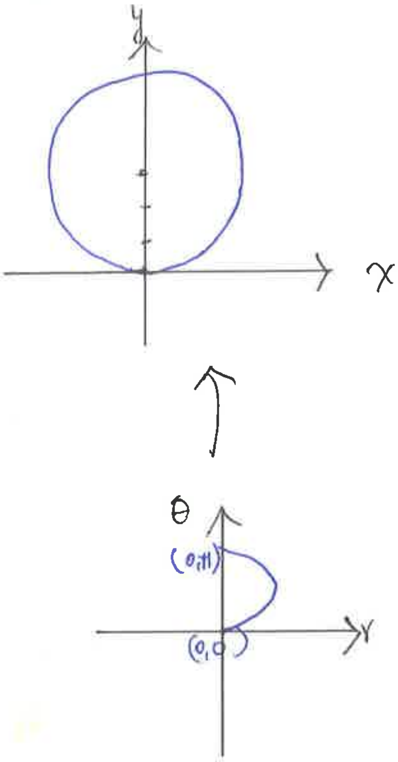




$y = x \Rightarrow r\sin\theta = r\cos\theta$

$$\Rightarrow \begin{cases} r = 0 \\ \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \end{cases}$$

eg II 13:



$$x^2 + (y-3)^2 = 9$$

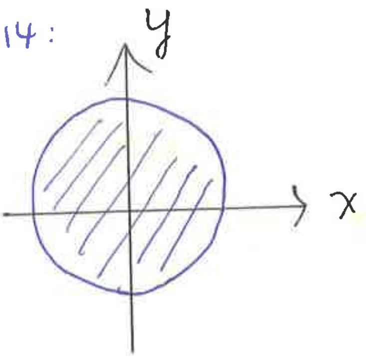
$$r^2\cos^2\theta + \{\sin\theta - 3\}^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + \cancel{9} = \cancel{9}$$

$$r^2 - 6r\sin\theta = 0$$

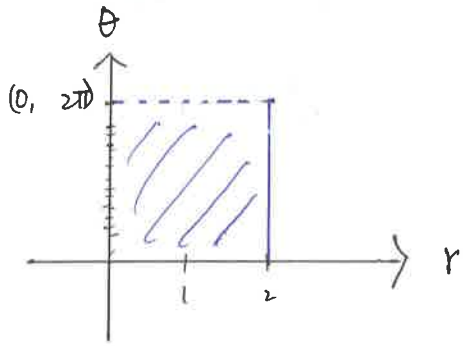
$$\Rightarrow r = 6\sin\theta \quad (\text{remember } r=0 \text{ excluded})$$

eg II 14:

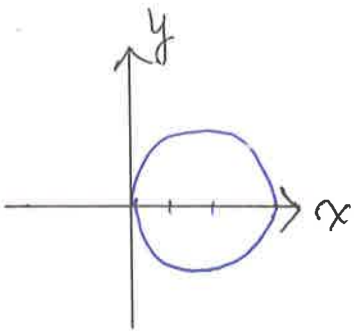


$$x^2 + y^2 \leq 4$$

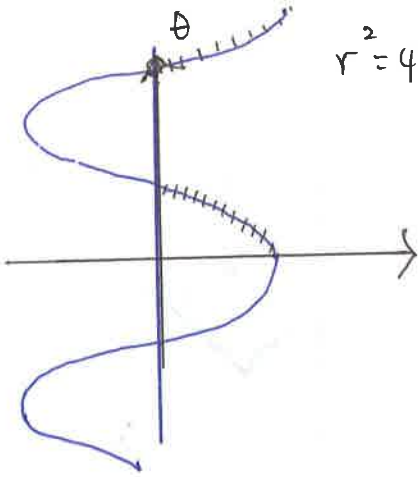
$$\xleftarrow{\Phi}$$



eg II 15.



$$\xleftarrow{\Phi}$$



$r^2 = 4r\cos\theta$

$r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 2^2$

eg II16

$$r = 4\tan\theta\sec\theta$$

$$= 4\,\frac{y}{x}\cdot\frac{1}{\cos\theta} \Rightarrow x = 4\,\frac{y}{x} \Rightarrow y = \frac{1}{4}x^2$$

eg II17

$$r = \csc\theta\, e^{r\cos\theta}$$

$$\frac{r}{\csc\theta} = e^{r\cos\theta} \Rightarrow r\sin\theta = e^{r\cos\theta}$$

OR $y = e^{x}$

eg II18

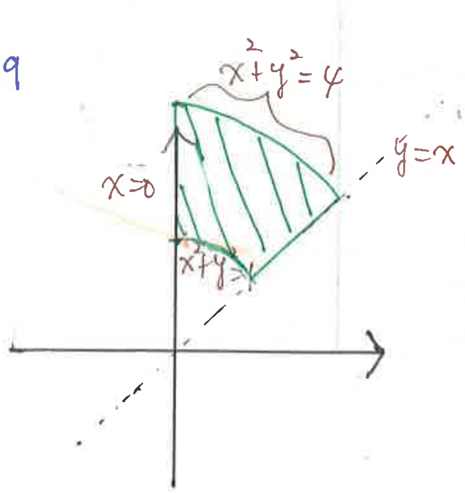
$$r = 8\sin\theta$$

$$r^2 = 8r\sin\theta \Rightarrow x^2 + y^2 = 8y$$

$$x^2 + y^2 - 8y + 16 = 16$$

$$x^2 + (y-4)^2 = 4^2$$

eg II19



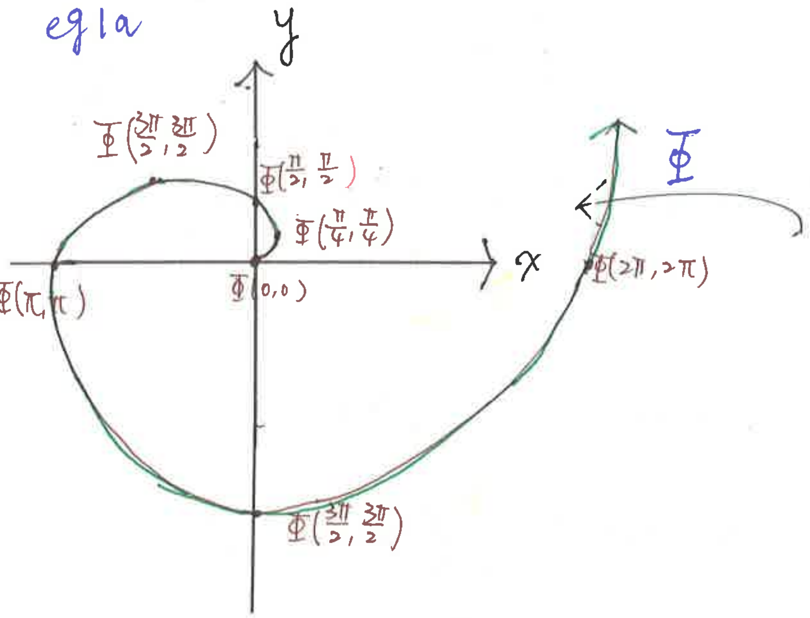
* More Interesting Curves via Φ

no more restriction on r, θ.

Forget about Φ as a coordinate and see some interesting curve on xy plane as images of Φ:

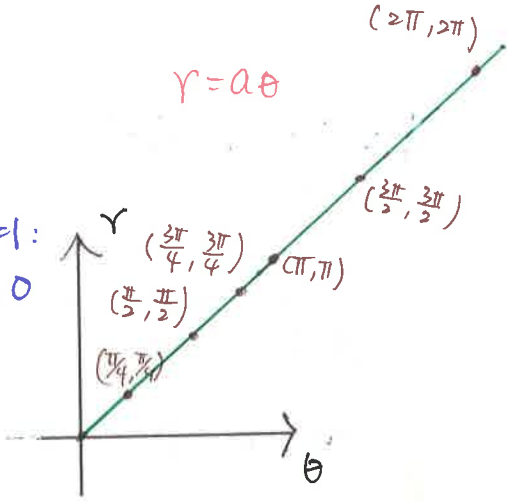
1. Spiral.

eg 1a



$r = a\theta$

$a=1$: $\theta \geq 0$



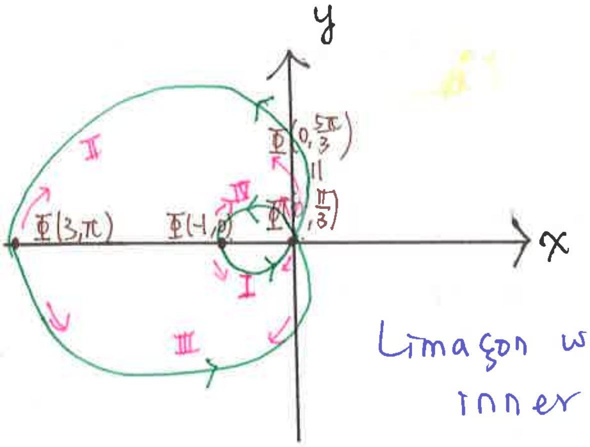
HW: Try $r = a\theta$, $a > 0$

2. Limaçon. (French for "snarl")

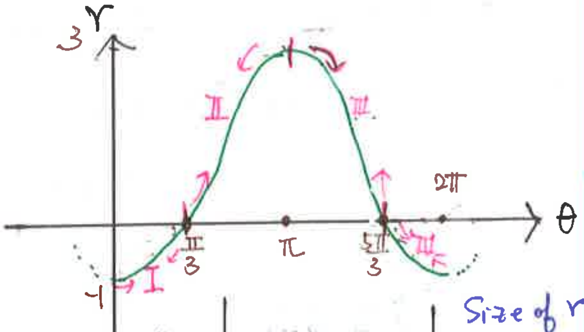
$$r = a + b\begin{cases}\sin\theta \\ \cos\theta\end{cases}$$

eg 2a.

$r = 1 - 2\cos\theta$



Φ



Limaçon with an inner loop

* r changes sign.

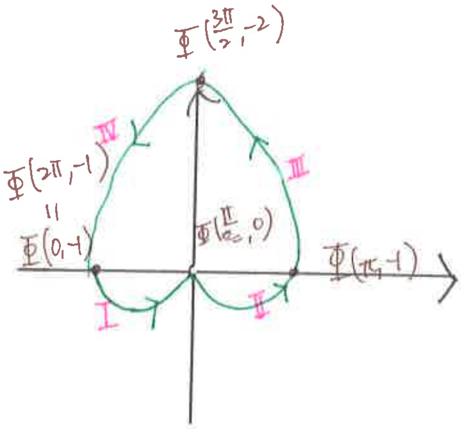
		Size of r
I	Y	↘
II	N	↗
III	N	↘
IV	Y	↗

eg 2b.

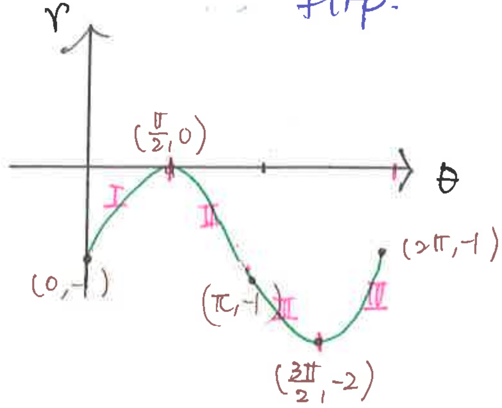
r never changes sign

$r = -1 + \sin\theta \leq 0$

flip.



$\xleftarrow{\Phi}$



In general

- $\left|\frac{a}{b}\right|$ and $|a|-|b|$ determine the geometry

Four possibilities :

1. limaçon w/ inner loop 
2. cardioids (Heart) 
3. Dimpled limaçon 
4. Oval (convex) limaçon 

H.W: Classify values of $\left|\frac{a}{b}\right|$ & $|a|-|b|$ in each possibility

- $\cos\theta$, $\sin\theta$ and sign of b determine the orientation of limaçon.

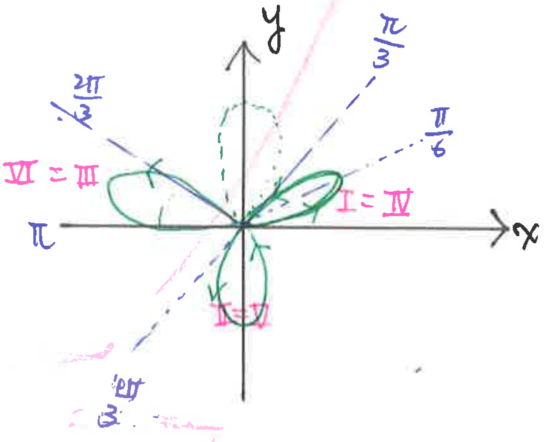
(e.g. whether dimple is at top? bottom? left? right?)

H.W: Discuss this!

3. Roses

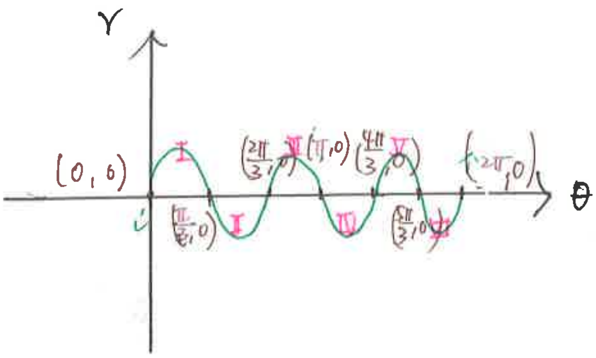
$$r = a\begin{cases}\sin(n\theta)\\ \cos(n\theta)\end{cases} ; \; n \in \mathbb{N}$$

eg 3a//

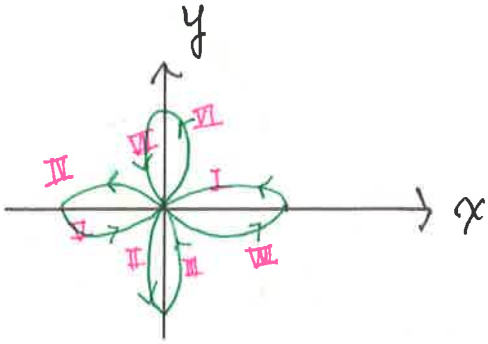


Φ

$r = \sin(3\theta)$

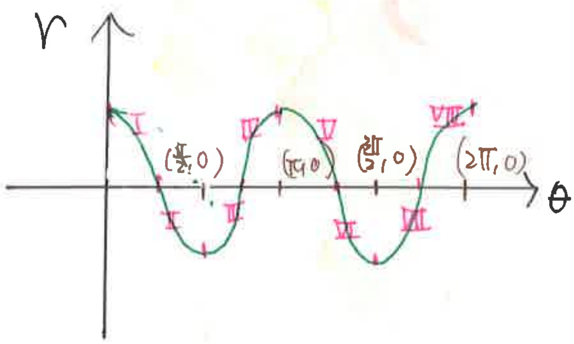


eg 3b//



Φ

$r = 2\cos(2\theta)$



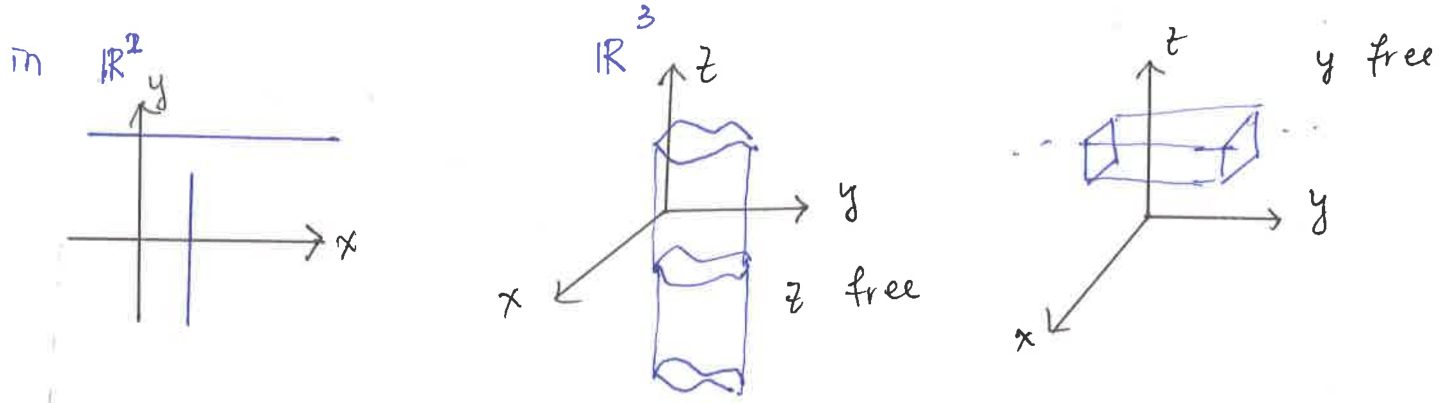
HW: Prove that, for both $r = a\sin(n\theta)$ & $r = a\cos(n\theta)$

n even $\Rightarrow$ $2n$ petals

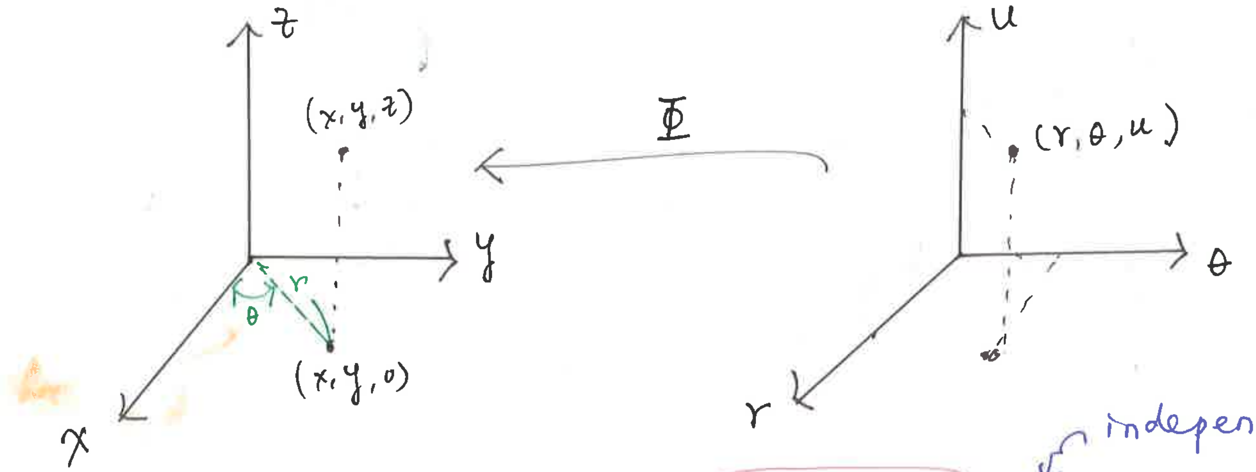
n odd $\Rightarrow$ n petals.

* Cylindrical Coordinates. System

"cylinder" in $\mathbb{R}^n$ is a trace where one variable is free (i.e. doesn't appear in the defining equation $P(\vec{x})$)



Cylindrical Coordinates



$$\Phi(r,\theta,u) = (\underbrace{r\cos\theta}_{x(r,\theta,u)},\ \underbrace{r\sin\theta}_{y(r,\theta,u)},\ \underbrace{u}_{z(r,\theta,u)})$$

independent of r, ~~θ θ~~

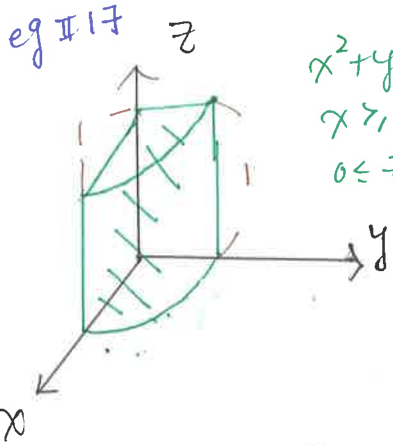
exactly the same as polar coordinates

didn't do anything

∴ just polar coordinate at each each z (u) - value

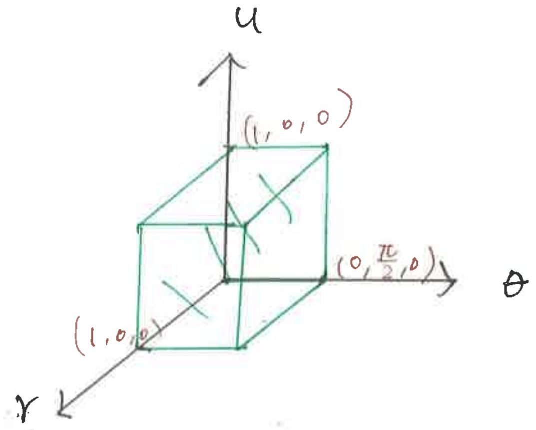
Most suitable for describing "cylindrical stuff":

eg II.17



$x^2+y^2 \le 1$
$x \geqslant 0,\ y \geqslant 0$
$0 \le z \le 1$

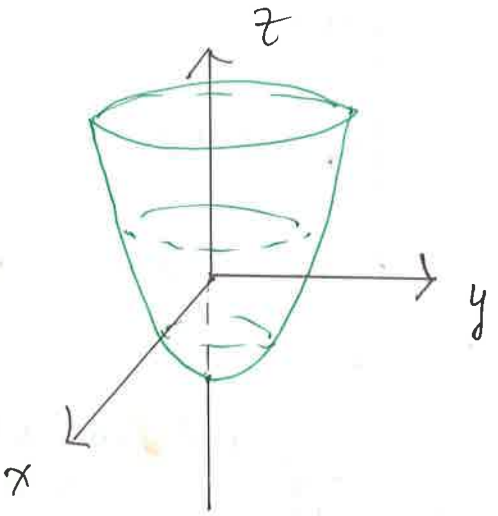
$\longleftarrow$

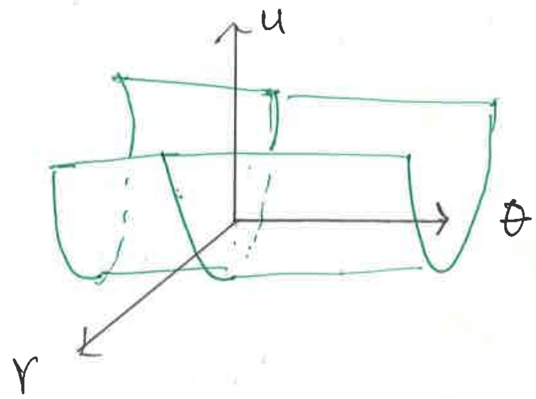


eg #18

$x^2+y^2-z=4 \quad \longleftarrow \quad r^2-u=4$

OR $\quad u = r^2-4$



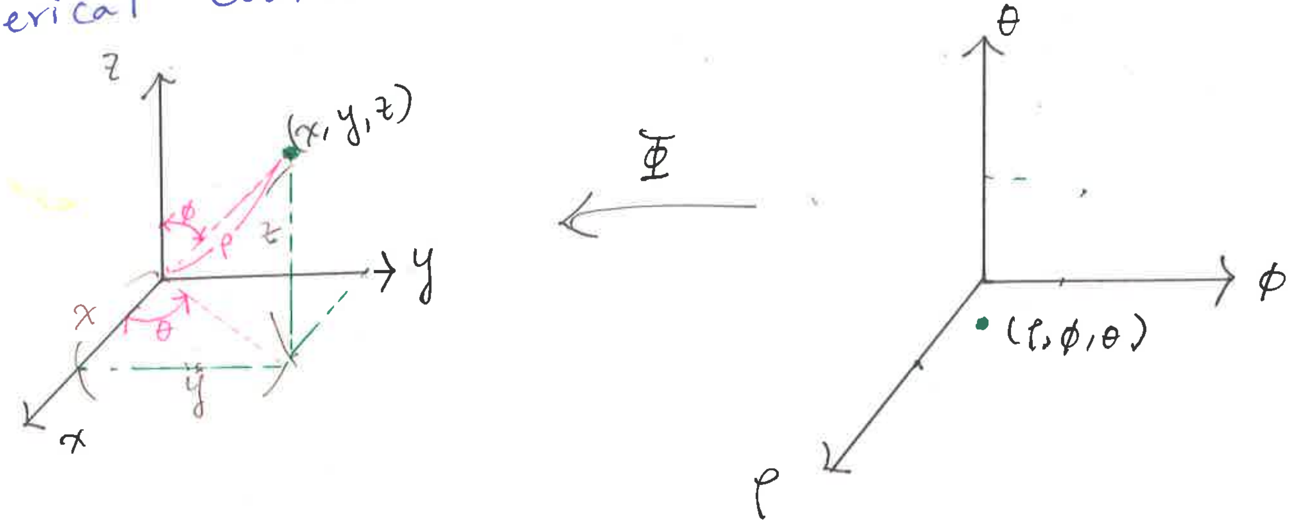


* Spherical Coordinates

Sphere in $\mathbb{R}^n$ w/ radius r centered at $\vec{a} = (a_1, \dots, a_n)$

$$= \{ (x_1, \dots, x_n) \mid (x_1 - a_1)^2 + \dots + (x_n - a_n)^2 = r^2 \}$$

Spherical Coordinates:



$$\Phi(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \ \rho \sin\phi \sin\theta, \ \rho \cos\phi)$$

Domain restriction: $\rho \in [0, \infty)$, $\theta \in [0, 2\pi)$.

(similar reason w/ polar coordinate)

$\phi \in [0, \pi]$, since for $\phi \in [\pi, 2\pi)$

$$\Phi(\rho, \phi, \theta) = \Phi(\rho, 2\pi - \phi, \theta \pm \pi)$$

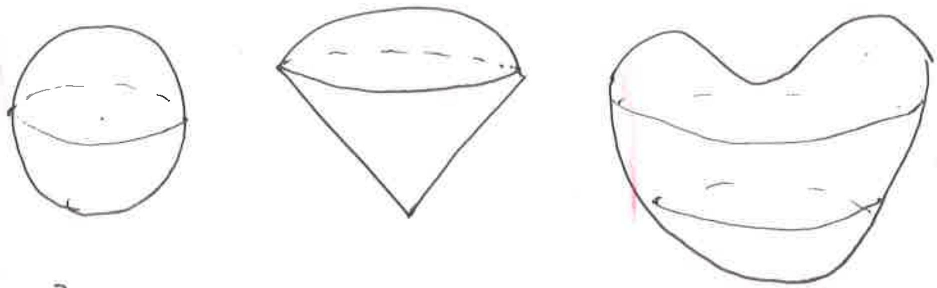
$\theta \in [0, \pi)$; $\theta \in [\pi, 2\pi)$

and the same rule applies to deal with origin.

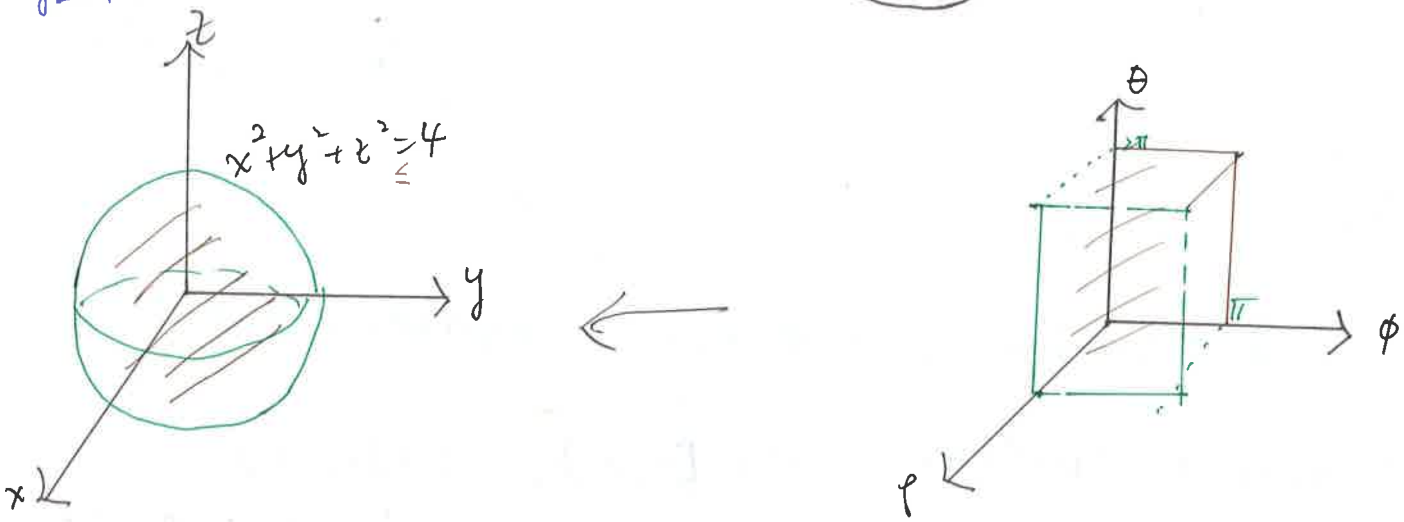
i.e. make sure Φ is one-to-one, for change of coordinate

Most suitable for describing objects w/ high "radial symmetric",

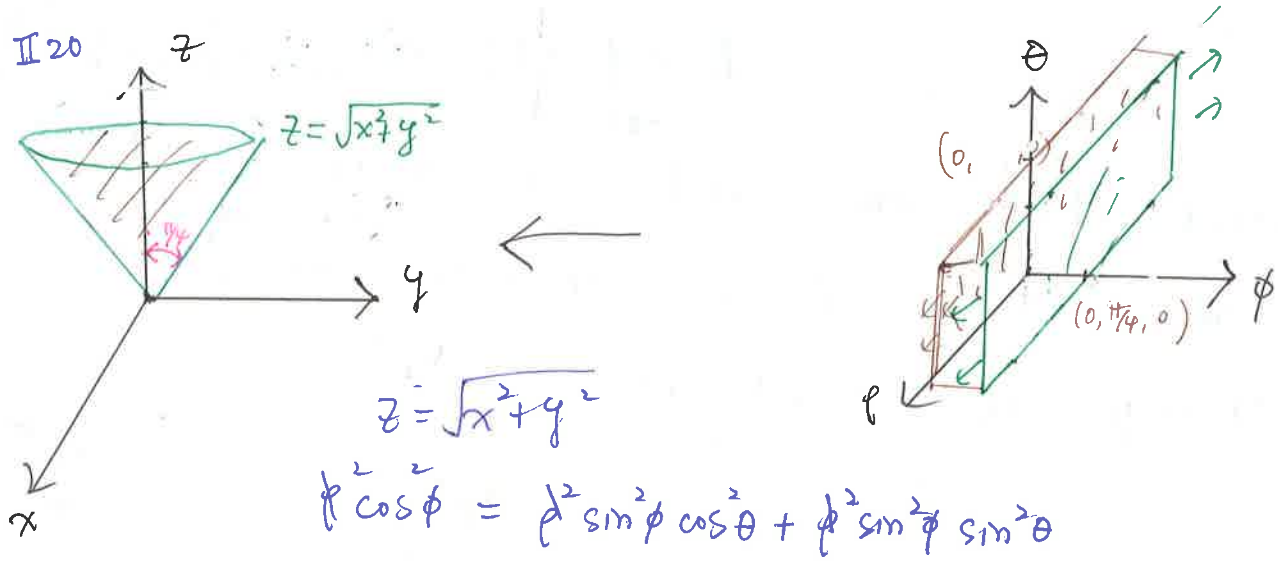
↳ dependent only on distance to origin.



eg II 19



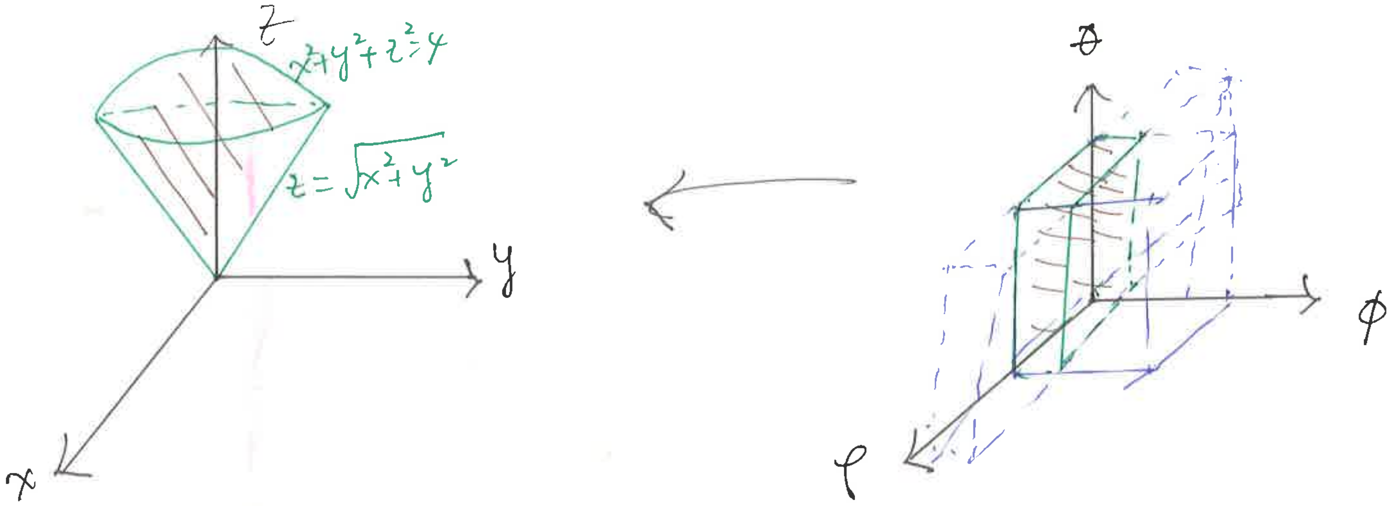
eg. II 20



$z=\sqrt{x^2+y^2}$

$\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$

$\rho\neq 0 \quad \cos^2\phi = \sin^2\phi \Rightarrow \phi=\frac{\pi}{4}$



z
$x^2+y^2+z^2=4$
$z=\sqrt{x^2+y^2}$
y
x
θ
ϕ
ρ