IV. Differentiation

Q. How fast is a particle moving on a line at any given time?

Intepretation of "How Fast"

in English, How fast at time x = How much is position $f(x)$ expected to change after one unit time past x.

"velocity".

position : is a function of time x, denoted $f(x)$

algebraically, "How fast" = rate of change of a function at x

a function changes at x

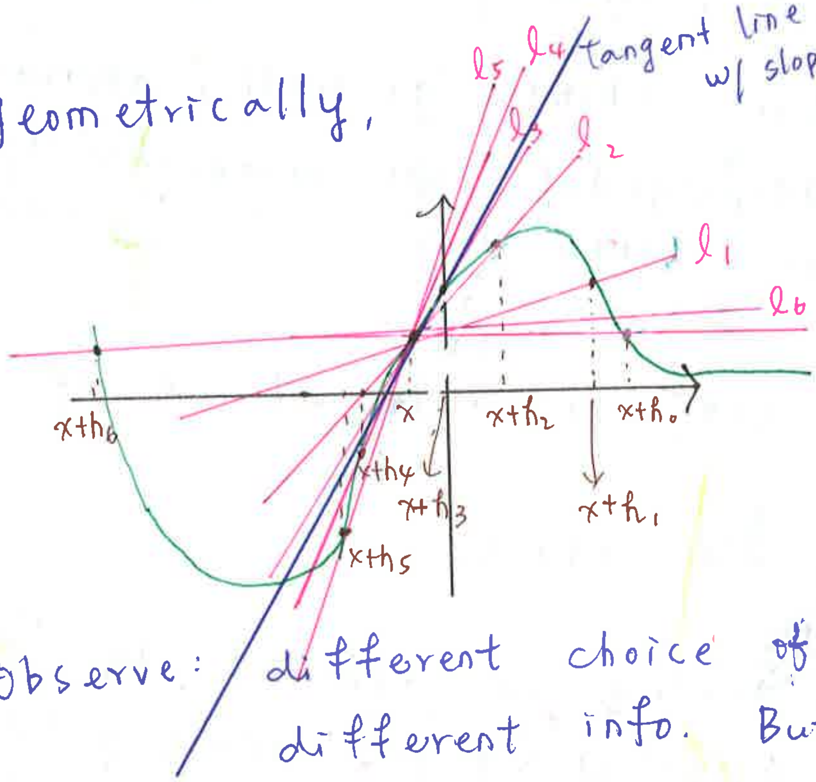
change in f due to change in x

in average sense

$$"=" \ \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

h ($\Delta x = h$)

geometrically,



$$\frac{f(x+h_i) - f(x)}{h_i} = \text{slope of "secant" line } l_i$$

explain what slope represents

$$\left(\downarrow h_i \to 0 \quad \text{"0"} \right)$$

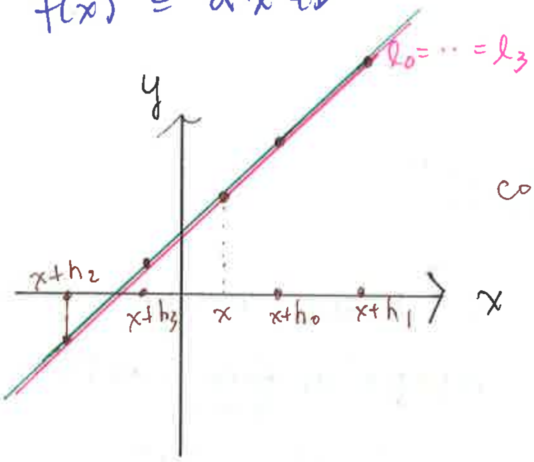
Observe: different choice of h_i (& so l_i) gives different info. But smaller (in abs. value) h_i generally gives closer info.

②

When is choice of h irrelevant?

A: when the graph of the function is a line

$f(x) = ax + b$



constant velocity motion

indeed,

$$\frac{f(x+h)-f(x)}{h} = \frac{a(x+h)+b-(ax+b)}{h} = a \quad \text{for all } x$$

and all secant lines precisely represent rate of change of $f(x)$

(average velocity)

Otherwise, slope of secant lines only approximate rate of change of f at x, and gets better as $|h|$ gets smaller, and when things go well, approaches exact value as $h \to 0$, $\to$ slope of tangent line. (instantaneous velocity)

導數

Def^n (Derivative)

A function f is differentiable at x if

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists.}$$

導函數 When existed, the value is denoted by $f'(x)$, $f': D \to \mathbb{R}$ is called the derivative of f, where $D = \{x \mid f \text{ differentiable at } x\}$

$x \mapsto f'(x)$. Also, $f \rightsquigarrow f'$ is called "differentiation" 微分

Geometrically, $f'(x)$ = slope of tangent line to $y = f(x)$ at x.

eg VI 1. $f(x) = ax + b \Rightarrow f'(x) = a$ for all x

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} a = a.$$

in particular, $\underbrace{a=0}_{f \text{ is constant}} \Rightarrow \underbrace{f'(x) = 0}_{f \text{ never changes (rate of change} \equiv 0)}$

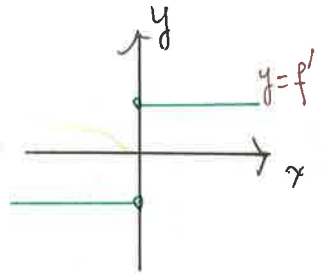
When do things go wrong?

eg VI 2. $\lim_{t\to x} f(t) \neq f(x)$ ie f not continuous at x

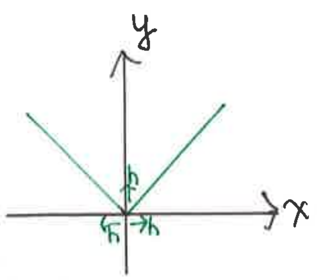
$$\Rightarrow \lim_{h\to 0} [f(x+h) - f(x)] > 0 \quad (\text{OR} < 0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ can't exist.}$$

continuity is not sufficient for differentiability!



eg VI 3. $f(x) = |x|$.



$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

$$\neq$$

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

$\Rightarrow$ $f(x)$ not differentiable at 0

However, differentiable at all other x

* Properties of Differentiations & Formulae

Differentiation is "linear" ← commutes w/ addition & scalar multiplication

Notations
$$(\alpha f)(x) = \alpha \cdot f(x) \qquad (fg)(x) = f(x)\,g(x)$$
$$(f+g)(x) = f(x)+g(x) \qquad \left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} \text{ if } f(x)\neq 0.$$

$$\Rightarrow \quad (\alpha_1 f_1 \pm \cdots \pm \alpha_n f_n)'(x) = \alpha_1 f_1'(x) \pm \cdots \pm \alpha_n f_n'(x)$$

This is true simply because $\lim(\alpha_1 f_1 \pm \cdots \pm \alpha_n f_n)$
$= \alpha_1 \lim f_1 \pm \cdots \pm \alpha_n \lim f_n$.

Note $\alpha_1 = 1$, $\alpha_2 = -1 \Rightarrow (f_1 - f_2)'(x) = f_1'(x) - f_2'(x)$.

eg III 7
$f(x) = 2x^2$
$\Rightarrow f'(x) = 2\cdot(x^2)'$
$= 4x$

eg III 8
$f(x) = x^3 + 4x^2$
$\Rightarrow f'(x) = 3x^2 + 8x$

However, differentiation does not commute w/ product

Product Rule

$$(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x) \quad \text{if } f, g \text{ diff. at } x$$

$$\xrightarrow{\text{induction}} (f_1 \cdots f_n)'(x) = f_1'(x) f_2(x) \cdots f_n(x) + f_1(x) f_2'(x) \cdots f_n(x) + \cdots + f_1(x) \cdots f_{n-1}(x) f_n'(x)$$

pf//

$$\lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x) f'(x) + f(x) g'(x)$$

//

eg IV 9 $\quad f(x) = x^{\frac{5}{2}} = x^2 \cdot x^{\frac{1}{2}} = x^2\sqrt{x}$

For $x>0$ $\quad f'(x) = (x^2)'\sqrt{x} + x^2(\sqrt{x})' = 2x\sqrt{x} + x^2 \cdot \frac{1}{2\sqrt{x}} = \frac{5}{2}\sqrt{x^3} = \frac{5}{2}x^{\frac{3}{2}}$

Reciprocal Rule

f diff. at x and $f(x) \neq 0$

$$\Rightarrow \left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$$

pf:

$$\lim_{h\to 0} \frac{\left(\frac{1}{f}\right)(x+h) - \left(\frac{1}{f}\right)(x)}{h} = \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{1}{f(x+h)} - \frac{1}{f(x)}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\ \frac{f(x)-f(x+h)}{f(x+h)f(x)}\right]$$

$$= \underbrace{\lim_{h\to 0}\frac{f(x)-f(x+h)}{h}}\ \underbrace{\lim_{h\to 0}\frac{1}{f(x+h)f(x)}}$$

$$= -f'(x) \cdot \frac{1}{[f(x)]^2} \quad //$$

eg. IV 10

$$g(x) = x^{-2} = \frac{1}{x^2} = \frac{1}{f}(x) \text{ where } f(x) = x^2$$

$$\Rightarrow g'(x) = -2x \cdot \frac{1}{x^4} \qquad (x \neq 0)$$

$$= -2x^{-3} \quad //$$

Quotient Rule

f, g diff. at x & $g(x) \neq 0$

$$\Rightarrow \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

pf: apply product & reciprocal rules

Derivative of Polynomials

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ (we've seen $n=2$ & 3)

for any integer n

$(\Rightarrow (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)' = n a_n \cdot x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1)$

Pf// $\underset{n \geq 0}{}$ Recall binomial expansion. $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + b^n$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

where $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\sum_{j=0}^{n} \binom{n}{j} x^{n-j} h^j - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + n x^{n-1} h + \sum_{j=2}^{n} \binom{n}{j} x^{n-j} h^j - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \Big(n x^{n-1} + \sum_{j=2}^{n} \underbrace{\binom{n}{j} x^{n-j} h^{j-1}}_{\downarrow \text{ as } h \to 0 \atop 0} \Big) = n x^{n-1} \quad //$$

$n < 0$: consider $f(x) = \dfrac{1}{x^{-n}}$; $-n > 0$

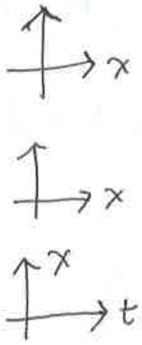
using reciprocal rule $f'(x) = \dfrac{-(-n x^{-n-1})}{x^{-2n}} = n x^{n-1}$

The formula actually works for ALL $n \in \mathbb{R}$.

* Lebiniz Differentiation Notations

- functions of derivatives (derivatives at general x)

	Newtonian Notation		Lebiniz Notation	
f a func of x	$f'(x)$		$\frac{df}{dx}$	OR $\frac{d}{dx}(f(x))$
y " "	$y'(x)$		$\frac{dy}{dx}$	suggest slope $\frac{\Delta y}{\Delta x}$
x " " t	$x'(t)$		$\frac{dx}{dt}$	$\frac{d}{dt}(x(t))$
	⋮		⋮	



- derivatives at specific point

Newtonian	Lebiniz
$f'(a)$	$\left.\frac{df}{dx}\right\vert_{x=a}$ (at)
$y'(a)$	$\left.\frac{dy}{dx}\right\vert_{x=a}$
$x'(a)$	$\left.\frac{dx}{dt}\right\vert_{t=a}$
⋮	⋮

(t) $x=a$

make more sense when there are more than one variables

eg4

$$f(x) = x^3 - 4x$$

$$\frac{df}{dx} = 3x^2 - 4 \qquad \left.\frac{df}{dx}\right\vert_{x=1} = -1$$

$$\Updownarrow \qquad \Updownarrow$$

$$\frac{d}{dx}(x^3 - 4x) = 3x^2 - 4 \qquad \left.\frac{d}{dx}\right\vert_{x=1}(x^3 - 4x) = -1$$

$$\Updownarrow \qquad \Updownarrow$$

$$f'(x) = 3x^2 - 4 \qquad f'(1) = -1$$

eg IV

$F(x) = (x^3-5x)\,g(x)$ & $g(2)=3$ & $g'(2)=-1$

$F'(2) = ?$

$F'(x) = (3x^2-5)\,g(x) + (x^3-5x)\,g'(x)$ (product rule)

$F'(2) = 7\,g(2) - 2\,g'(2) = 21+2 = 23$ //

Warning! This is wrong

$F'(2) = (3x^2-5)\big|_{x=2}\, g(2) + (x^3-5x)\,[g(2)]' \to 0$

$= 7\,g(2) = 21.$ ✗

can't plug in any specific value after all the derivatives are taken, as derivative at x involve <u>all points</u> near x, not just x

eg V

$F(x) = \dfrac{6x^2-1}{x^4+5x+1}$

using quotient rule

$$F'(x) = \frac{12x(x^4+5x+1)-(4x^3+5)(12x)}{(x^4+5x+1)^2}$$ //

eg11 $u(t) = t(t+1)(t+2)$

$$\frac{du}{dt} = (t+1)(t+2) + t(t+2) + t(t+1)$$

$$\Updownarrow$$

$$\frac{d}{dt}\left[t(t+1)(t+2)\right] = t+1)(t+2) + t(t+2) + t(t+1)$$

$$\Updownarrow$$

$$u'(t) = (t+1)(t+2) + t(t+2) + t(t+1)$$

$$\left.\frac{du}{dt}\right|_{t=0} = 2 \iff \left.\frac{d}{dt}\right|_{t=0}\left[t(t+1)(t+2)\right] = 2 \iff u'(0) = 2.$$

eg.

$$f(x) = \begin{cases} x^2 ; & x \geq 0 \\ x^3 ; & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x ; & x > 0 \\ 0 ; & x = 0 \\ 3x^2 ; & x < 0 \end{cases}$$

$$f''(x) \text{ "=" } \begin{cases} 2 ; & x > 0 \\ ? & x = 0 \\ 6x ; & x < 0 \end{cases}$$

($ax+b$ w/ $a=2$ & $b=0$)

(linearity)

compute

$$\lim_{h \to 0} \frac{f'(h) - f'(0)}{h} \quad *$$

$h \to 0^+$; $* = \lim_{h \to 0^+} \frac{2h - 0}{h} = 2$

$\neq$

$h \to 0^-$ $* = \lim_{h \to 0^-} \frac{h^2 - 0}{h} = 0$

$\therefore$ $f''(0)$ DNE.

(indeed, there is a jump.)

$\therefore$ f is C^1 but not C^2 on $\mathbb{R}$ (and certainly not any higher C^k)

However f is $C^\infty(U)$ for any connected $U \subset \mathbb{R}$ not containing 0.

Observe : $C^k(U) \subset C^l(U)$ for all $k \leq l$

$C^0(U)$: all continuous functions.

* Derivatives of Higher Order.

Given $f: D_0 \to \mathbb{R}$, continuous we differentiate it to get

$f': D_1 \to \mathbb{R}$; $D_1 = \{x \in D_0 \mid \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists}\}$

$\left(\frac{df}{dx}\right)$ $\left(\frac{d}{dx}(\cdots)\right)$ to differentiate f' to get

f'' $\left(\frac{d^2f}{dx^2}\right)$ $\left(\frac{d^2}{dx^2}(\cdots)\right)$: $D_2 \to \mathbb{R}$; $D_2 = \{x \in D_1 \mid \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} \text{ exists}\}$

⋮ keep differentiating

$f^{(n)}$ $\left(\frac{d^nf}{dx^n}\right)$ $\left(\frac{d^n}{dx^n}(\cdots)\right)$: $D_n \to \mathbb{R}$; $D_n = \{x \in D_{n-1} \mid \lim_{h\to 0} \frac{f^{(n-1)}(x+h)-f^{(n-1)}(x)}{h} \text{ exists}\}$

rate of change
derivative of f of order n ; "n^{TH} derivative" of f with respect to x
with respect to x rate of change

For $U \subset \mathbb{R}$, $C^k(U) := \{f: U \to \mathbb{R} \mid f^{(j)} \text{ exists on } U;\ 1 \le j \le k\}$

"f is C^k"

if $k = \infty$; f is said to be "smooth" on U.

eg., $f(x) = x^2$ is smooth on $\mathbb{R}$

$f'(x) = 2x$; $f''(x) = 2$; $f^{(n)}(x) = 0$ for all $n \geq 3$.

$\left(\frac{df}{dx} = 2x \ \left(\frac{d}{dx}(x^2) = 2x\right)\right.$ & $\frac{d^nf}{dx^n} = 0$ for all $n \geq 3$)

All "nice functions" are smooth except $\frac{\sim}{0}$.

eg

$$f(x) = (x^2+2)(x^{-2}+2)$$

$$f'(x) = 2x(x^{-2}+2) + (x^2+2)(-2x^{-3})$$

$$= \cancel{\frac{2}{x}} + 4x - \cancel{\frac{2}{x}} - 4x^{-3}$$

$$f''(x) = 4 + 12x^{-4}$$

$$= 4 + \frac{12}{x^4}.$$

eg

$$\frac{d^5}{dt^5}(at^4 + bt^3 + dt^2 + et + f) = 0$$

eg

$$\frac{d}{du}\left[u\frac{d}{du}(u-u^2)\right] = \frac{d}{du}(u-u^2) + u\frac{d^2}{du^2}(u-u^2)$$

$$\left([u(u-u^2)']'\right)$$

$$= (1-2u) + u(-2) = 1-2u-2u = 1-4u$$

* Rate of change

Recall: Given f as a function of x.

$f'(x) = \frac{df}{dx}$ = rate of change of f with (wrt) respect to x

= expected change of f when $x \mapsto x+1$

= slope of tangent line to $y = f(x)$

$f'(a) = \left.\frac{df}{dx}\right|_{x=a}$ = " " " " " at a

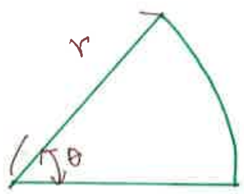
= " " " " " $x=a \mapsto x=a+1$

= " " " " at $x=a$.

more commonly used as it reminds us slope $\frac{\Delta y}{\Delta x}$.

x & f are just notations & may be replaced in different situations.

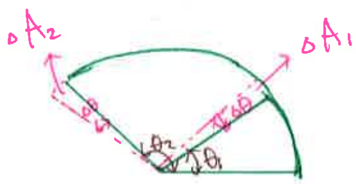
eg 11 (Area of sector)



$A(r,\theta) = \frac{1}{2}r^2\theta$

if r is constant, $A(\theta) = \frac{1}{2}r^2\theta$

and $\frac{dA}{d\theta} = \frac{1}{2}r^2$ for all θ.



$\Delta A_1 = \Delta A_2$

ie. $\frac{dA}{d\theta}$ independent of θ.

A changes equally fast at all angle.

if θ is constant

$\Delta A_3 > \Delta A_2 > \Delta A_1$

$A(r) = \frac{1}{2}r^2\theta$ & $\frac{dA}{dr} = 2r$

A changes faster (slower) at larger (smaller) r if A is constant:

$$\theta(r) = \frac{2A}{r^2} \quad \& \quad \frac{d\theta}{dr} = -\frac{4A}{r^3}$$

$$r(\theta) = \frac{\sqrt{2A}}{\sqrt{\theta}} = \sqrt{2A}\,\theta^{-\frac{1}{2}} \quad \& \quad \frac{dr}{d\theta} = -\sqrt{\frac{A}{2}}\,\theta^{-\frac{3}{2}} = \frac{-\sqrt{A/2}}{\sqrt{\theta^3}}$$

θ decreases as r increases, & faster as r gets larger

r " " θ " & faster " θ " " (but not as fast as $\frac{d\theta}{dr}$)

* Chain Rule

$\frac{d}{dx} f(g(x)) = ?$ (particularly when we don't have a clear formula like $\sqrt{x^2+4}$, $\cos(e^{x^2+1})$ ...)

It's easy if we can "unwind" $f(g(x))$.

eg. $f(u) = u^2+1$ & $g(x) = 2x+1$

$\Rightarrow f(g(x)) = (2x+1)^2+1 = 4x^2+4x+2$

and $\frac{d}{dx} f(g(x)) = 8x+4$

But what about

$f(u) = \sqrt{u}$ and $g(x) = x^3+7$

$\Rightarrow f(g(x)) = \sqrt{x^3+7}$?

$f(u) = u^3+2$ and $g(x) = \sqrt{x+1}$

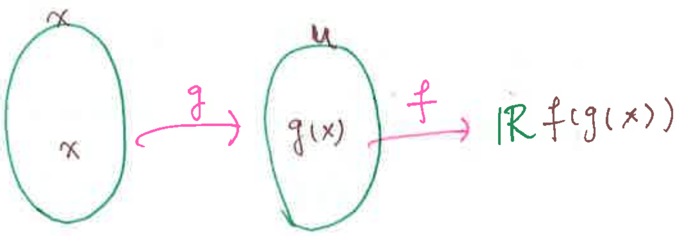
$\Rightarrow f(g(x)) = (\sqrt{x+1})^3+2$?

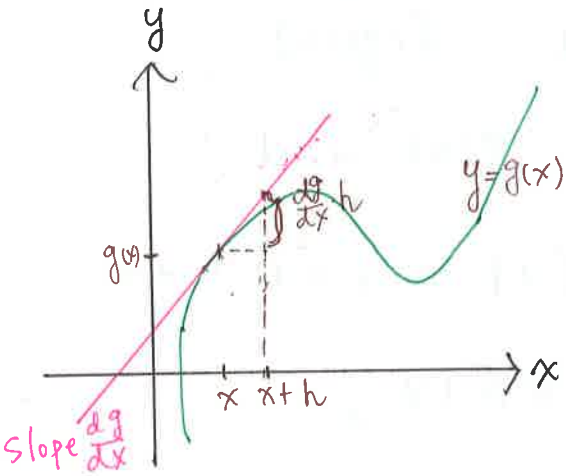
Need to develop some machineries!

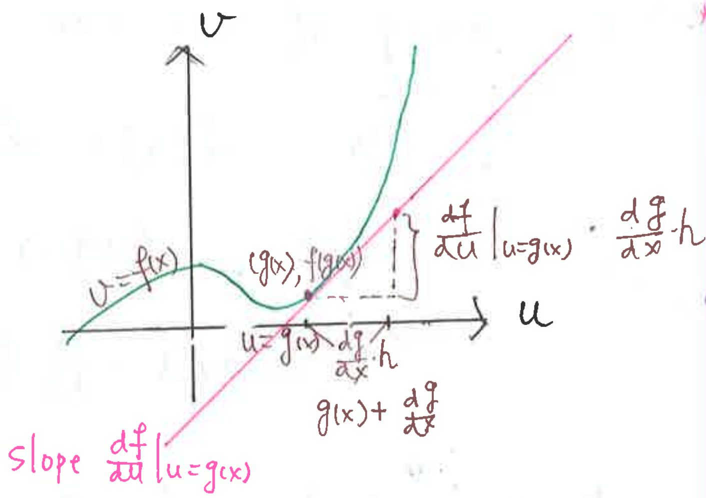
Recall: on a line w/ slope x))

m, if $x \to x+1$, $y \to y+m$

Intuitive Derivation :







$$\frac{d}{dx} f(g(x)) = \text{expected change of } f \text{ when } x \to x+h$$

$$x \mapsto x+1 \rightsquigarrow g(x) \to (\approx)\ g(x) + \frac{dg}{dx} h$$

$$f(g(x)) \to (\approx)\ f(g(x)) + \underbrace{\left.\frac{df}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx} h}_{\frac{d}{dx} f(g(x))}$$

$$\therefore \lim_{h \to 0} \frac{f(g(x+h)) - f(g(h))}{h}$$

$$= \lim_{h \to 0} \frac{\left.\frac{df}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx} \cdot h}{h}$$

$$= \left.\frac{df}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx}$$

Thm. $\frac{d}{dx}(f \circ g) = \left.\frac{df}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx}$ (OR $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$)

(Given $f(u)$ and $g(x)$)

pf.

$$\lim_{h \to 0} \frac{f \circ g(x+h) - f \circ g(x)}{h} = ?$$

$$g(x+h) = g(x) + h g'(x) + E(h)$$

$$\& \quad \lim_{h \to 0} E(h) = \lim_{h \to 0} \frac{E(h)}{h} = 0$$

$$f(u+p) = f(u) + p f'(u) + F(p)$$

$$\& \quad \lim_{p \to 0} F(p) = \lim_{p \to 0} \frac{F(p)}{p} = 0$$

$$\therefore \ f(g(x+h)) = f(\overbrace{g(x)}^{u} + \overbrace{h g'(x) + E(h)}^{p})$$

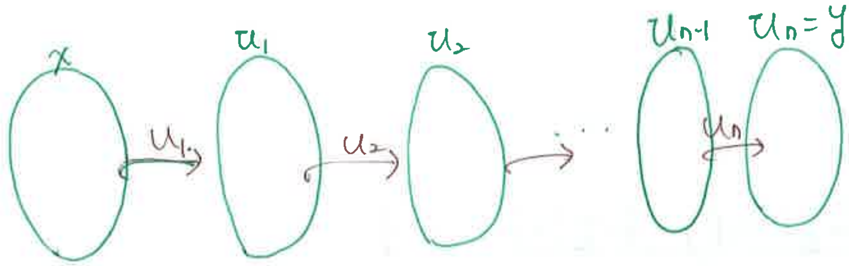
$$= f(g(x)) + [h g'(x) + E(h)] f'(g(x)) + F(h g'(x) + E(h))$$

$$\therefore \ \frac{f \circ g(x+h) - f \circ g(x)}{h}$$

$$= f'(g(x)) g'(x) + \underbrace{\frac{E(h)}{h}}_{\downarrow h \to 0 \ \ 0} + \underbrace{\frac{F(h g'(x) + E(h))}{h}}_{\downarrow h \to 0 \ \ 0}$$

#

Clearly, derivatives of multi-layer compositions are :



$$y = u_n(x) = u_n \circ \cdots \circ u_3 \circ u_2 \circ u_1(x)$$
$$= u_n(\cdots(u_3(u_2(u_1(x))))\cdots)$$

d : $$= \frac{dy}{dx} = \frac{du_n}{du_{n-1}}\bigg|_{u_{n-1}\circ\cdots\circ u_1(x)} \cdot \frac{du_{n-1}}{du_{n-2}}\bigg|_{u_{n-2}\circ\cdots\circ u_1(x)} \cdot \frac{du_{n-2}}{du_{n-3}}\bigg|\cdots \cdots \frac{du_1}{dx}$$

eg 11 $$\frac{d}{dx}\left[\left(x+\frac{1}{x}\right)^{-3}\right] = ?$$

$$\left(x+\frac{1}{x}\right)^{-3} = f \circ g(x)$$ where

$$g(x) = x + \frac{1}{x} \quad \& \quad f(u) = u^{-3}$$

$$\frac{d}{dx}\left(f\circ g(x)\right) = \frac{df}{du}\bigg|_{u=g(x)} \frac{dg}{dx}$$
$$= -4\,u^{-4}\Big|_{u=x+\frac{1}{x}} \cdot \left(1-\frac{1}{x^2}\right)$$
$$= -3\left(x+\frac{1}{x}\right)^{-4}\left(1-\frac{1}{x^2}\right)$$

//

eg.

$$\frac{d}{dx}\left[2x^3(x^2-3)^4\right]$$

$$= \left[\frac{d}{dx}(2x^3)\right](x^2-3)^4 + 2x^3 \cdot \frac{d}{dx}\left[(x^2-3)^4\right]$$

$$= 6x^2(x^2-3)^4 + 2x^3 \frac{d}{dx}\left[f \circ g(x)\right]$$

where
$g(x) = x^2-3$
$f(u) = u^4$

$$= 6x^2(x^2-3)^4 + 2x^3 \cdot 4u^3\Big|_{u=x^2-3} \cdot (2x)$$

$$= 6x^2(x^2-3)^4 + 4(x^2-3) \cdot 8x^4$$

//

$(x^2-3)^4 \to 4(x^2-3)^3$

differentiate the entire thing

then multiply by derivative inside (i.e. $2x$)

eg. (Multi-layer)

$$\frac{d}{dx}\left[(\sqrt{x}+1)^2+3x\right]^4$$

$$= 4\left[(\sqrt{x}+1)^2+3x\right]^3 \cdot \frac{d}{dx}\left[(\sqrt{x}+1)^2+3x\right]$$

$$= 4\left[(\sqrt{x}+1)^2+3x\right]^3 \cdot \left[2(\sqrt{x}+1)\cdot\frac{1}{2\sqrt{x}}+3\right]$$

$$= 4\left[(\sqrt{x}+1)^2+3x\right]^3\left[4+\frac{1}{\sqrt{x}}\right]$$

//

~~10/15 ends~~ 10/17 ends

eg11 area inside a circle

$A(r) = \pi r^2$

if r is a function of time, $r(t)$, $\Rightarrow$

A is a function of time $A(r(t))$.

of A wrt time, $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$= 2\pi r(t) \cdot r'(t)$

$r(t) = 8t$, then $\left.\frac{dA}{dt}\right|_{t=1} = 2\pi r(1) \cdot r'(1)$

$= 16\pi \cdot 8 = 128\pi$ //

* Differentiation of Trigonometric Functions.

Thm//

$$\frac{d}{dx}(\sin x) = \cos x$$

Pf//

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

Recall $\sin(A+B)$
$= \sin A\cos B + \cos A\sin B$

$$= \lim_{h\to 0}\sin x\left(\frac{\cos h-1}{h}\right) + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

$h\to 0$, $\to 0$; $h\to 0$, $\to 1$

$$= \cos x$$

Thm//

$$\frac{d}{dx}\cos x = -\sin x$$

Pf//

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$= \cos x\lim_{h\to 0}\frac{\cos h-1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$h\to 0$, $\to 0$; $h\to 0$, $\to 1$

Recall:
$\cos(A+B)$
$= \cos A\cos B - \sin A\sin B$

$$= -\sin x$$

verify these two identities w/ graphes

Cor//

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\cdot\cos x+\sin x\sin x}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$
$$= \tan x\sec x.$$

Similarly, $\frac{d}{dx}\cot x = -\csc^2 x$ & $\frac{d}{dx}\csc x = -\cot x\csc x$ //

eg11 $\frac{d}{dt}\cos^2 t = 2\cos t \cdot \frac{d}{dt}\cos t = 2\cos t\,(-\sin t) = -2\cos t \sin t$

$= -\sin(2t)$

eg11 $\frac{d}{dx}\sec(x^2+1) = \sec(x^2+1)\tan(x^2+1)\cdot\frac{d}{dx}(x^2+1)$

$= 2x\sec(x^2+1)\tan(x^2+1)$

eg11 Find x in $[0, 2\pi]$ so that tangent line to

$y = \sin x + \sqrt{3}\cos x$

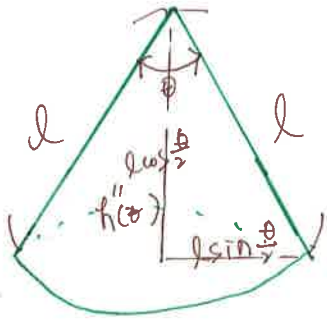
is horizontal

slope $= 0$

$\frac{dy}{dx} = \cos x - \sqrt{3}\sin x = 0 \Rightarrow \tan x = \frac{1}{\sqrt{3}}$

$\Rightarrow x = \frac{\pi}{6},\ \frac{7\pi}{6}$

eg. consider a cone formed by rotating two sticks of fixed length (but various angle θ)



let $V(\theta) = \frac{1}{3} A(\theta) h(\theta)$ be volume of the cone.

Find $\frac{dV}{d\theta}$.

$$A(\theta) = \pi l^2 \sin^2 \frac{\theta}{2}, \quad h(\theta) = l \cos\frac{\theta}{2}$$

$$V(\theta) = \pi l^3 \sin^2\frac{\theta}{2} \cos\frac{\theta}{2} = \pi l^3 \left[\sin\frac{\theta}{2}\right]^2 \cos\frac{\theta}{2}$$

$$\frac{dV}{d\theta} = \pi l^3 \cdot 2\left(\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\right)\cdot\left(\frac{1}{2}\right)\cos\frac{\theta}{2} - \frac{\pi l^3}{2}\left[\sin\frac{\theta}{2}\right]^2 \sin\frac{\theta}{2}$$

$$= \pi l^3 \sin\frac{\theta}{2}\left[\cos^2\frac{\theta}{2} - \frac{1}{2}\sin^2\frac{\theta}{2}\right]$$

if $\theta = \theta(t)$, a function of t, $\frac{dA}{dt} = ?$
$= t^2+2$

$$\frac{dV}{dt} = \left.\frac{dV}{d\theta}\right|_{\theta(t)} \cdot \frac{d\theta}{dt}$$

$$= \pi l^3 \sin\frac{t^2+2}{2}\left[\cos^2\frac{t^2+2}{2} - \frac{1}{2}\sin^2\frac{t^2+2}{2}\right]\cdot(2t)$$

//