以下為針對本學期最後一個主題”Change of Basis & Diagonalization”講義的修正。主要修正在地 3，4，5，8 頁，皆有用中文做註記，請同學以此版本為主將上課講義予以修正。

上課時本人不小心弄反了座標轉換矩陣的相對位置，其中 Q 應為將想要轉換的 basis (beta prime) 寫成初始給定 basis (beta)的座標表示法，但上課時將其誤植為 Q 的反矩陣。因此在對角化的過程中，若給定 basis(beta)為標準基底，而 beta prime 為 eigenbasis(若有的話)，則 Q 的每一行即為 eigenbasis 的各個向量(按照 eigenvalue 相對應的順序寫下)。

請將筆記再複習一遍，並釐清觀念，雖然整個對角化的概念之介紹沒有錯誤，但此技術性的失誤造成同學混淆本人深感抱歉。

XI. Change of Basis, Diagonalization

Motivation: Given want to write down an expression for

$$T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$$

$$\Updownarrow \text{ (fixing basis)}$$

Given $A \in \mathrm{Mat}_{n\times n}$, compute $\underbrace{A^m = A \cdots A}_{\text{can be very troublesome}}$

computing A^m can be troublesome, but there are some special convenient cases.

eg., when A is "diagonal", i.e. $a_{ij} \neq 0 \Rightarrow i = j$

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} := \mathrm{diag.}(\lambda_1, \ldots, \lambda_n)$$

$$\rightsquigarrow \quad A^m = \mathrm{diag.}(a_1^m, \ldots, a_n^m)$$

Q: Given $T: (\mathbb{R}^n, \beta) \to (\mathbb{R}^n, \beta)$. can (& how) we find an alternative basis γ so that

$$T: (\mathbb{R}^n, \gamma) \to (\mathbb{R}^n, \gamma)$$

has matrix $[T]_\gamma^\gamma = \mathrm{diag.}(\lambda_1, \ldots, \lambda_n)$?

If we can, what is the relationship between $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$?

An elementary example:

on $\mathbb{R}^2$, $\quad \beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$T : (\mathbb{R}^2, \beta) \to (\mathbb{R}^2, \beta)$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x + y \end{pmatrix} \quad ; \quad [T]_\beta^\beta = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

If we choose $\gamma = \{\gamma_1, \gamma_2\}$ so that

$$(\gamma_1)_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (\gamma_2)_\beta = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$$

$$\rightsquigarrow \quad \left(T(\gamma_1)\right)_\beta = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_\beta - \left(T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_\beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

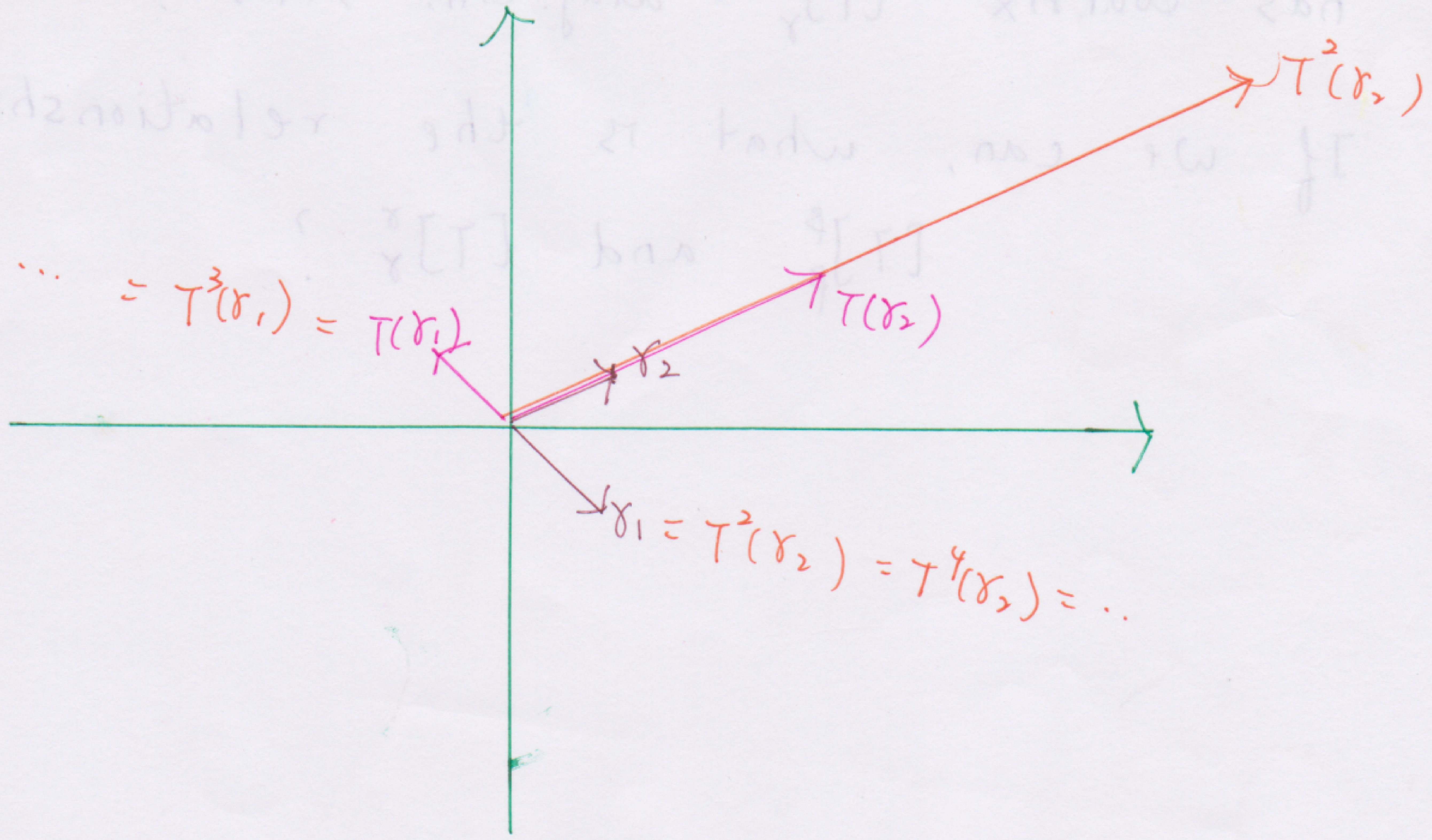
or, without coordinate. $T(\gamma_1) = -\gamma_1$.

$$\rightsquigarrow \quad \left(T(\gamma_2)\right)_\beta = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_\beta + \frac{2}{3}\left(T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_\beta = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{8}{3} \end{pmatrix}$$

$$= 4 \cdot \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

" $T(\gamma_2) = 4\gamma_2$.

We conclude: $\quad [T]_\gamma^\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$ and $\left([T]_\gamma^\gamma\right)^m = \begin{pmatrix} (-1)^m & 0 \\ 0 & 4^m \end{pmatrix}$



* Choice of Basis vs. Matrix Representation

Given $T: V \to W$, choose two sets of bases

$\beta = \{\beta_j\}_{j=1}^n$, $\beta' = \{\beta_j'\}_{j=1}^n$ for V and

$\gamma = \{\gamma_i\}_{i=1}^m$, $\gamma' = \{\gamma_i'\}_{i=1}^m$ for W.

What is the relationship between $[T]_\beta^\gamma$ and $[T]_{\beta'}^{\gamma'}$?

$$\begin{tikzcd}
(V,\beta) \arrow[r, "T"] & (W,\gamma) \\
(V,\beta') \arrow[u, "I_V"] \arrow[r, "T"] & (W,\gamma') \arrow[u, "I_W"']
\end{tikzcd}$$

As transformation, $T = I_W \circ T \circ I_V^{-1}$

As matrices $[T]_\beta^\gamma = [I_W \circ T \circ I_V^{-1}]_\beta^\gamma$

$$= [I_W]_{\gamma'}^{\gamma}[T]_{\beta'}^{\gamma'}\left([I_V]_{\beta'}^{\beta}\right)^{-1}$$

$$\Rightarrow [T]_{\beta'}^{\gamma'} = \left([I_W]_{\gamma'}^{\gamma}\right)^{-1}[T]_\beta^\gamma\,[I_V]_{\beta'}^{\beta} \quad \cdots \text{④}$$

有修改請對照

change of coordinate matrices

加註

i.e. $\beta = \{\beta_1, \dots, \beta_n\}$, $\beta' = \{\beta_1', \dots, \beta_n'\}$

$\Rightarrow [I_V]_{\beta'}^{\beta} = \left((\beta_1')_\beta, \dots, (\beta_n')_\beta \right)$; column j = coordinate representation of β_j' in basis β

We have the correspondence

上課未提可忽略

$$\mathcal{L}(V,W) \longleftrightarrow \mathrm{Mat}_{m\times n}/\sim \qquad \text{"equivalent class"}$$

$A \sim B \Leftrightarrow A = QBP$,

an equivalent relation, since

① $A = I_m A I_n \Rightarrow A \sim A$

② $A \sim B \Rightarrow A = QBP \Rightarrow B = Q^{-1}AP^{-1} \Rightarrow B \sim A$

③ $A \sim B$ & $B \sim C \Rightarrow A = QBP$ & $B = Q'CP'$
$\Rightarrow A = QQ'CP'P \Rightarrow A \sim C$

In particular, we discuss $V = W$ & $\beta = \gamma$.

(*) becomes

$$[T]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'} \, [T]_{\beta}^{\beta} \, [I_V]_{\beta'}^{\beta}$$
$$= \left([I_V]_{\beta'}^{\beta}\right)^{-1} [T]_{\beta}^{\beta} \, [I_V]_{\beta'}^{\beta}$$

Two $n \times n$ matrices A, B are called similar if

$$B = \underbrace{Q^{-1} A Q}_{A \text{ conjugated with } Q}$$

Two similar matrices represent the same linear transformation, with appropriate choices of basis:

請對照修改

ie. $B = Q^{-1} A Q$

If $A = [T]_{\beta}^{\beta} \Rightarrow B = [T]_{\beta'}^{\beta'}$, where $Q = [I_V]_{\beta'}^{\beta}$

and we have the correspondence

$$\mathcal{L}(V, V) \longleftrightarrow Mat_{n \times n}/\sim$$

$A \sim B \Leftrightarrow A$ is similar to B.

Back to the elementary example.

$\beta = \left\{ \underset{\beta_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{\beta_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$, $\beta' = \left\{ \underset{\beta_1'}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}, \underset{\beta_2'}{\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}} \right\}$: clearly

加註 in coordinates $\{e_j\}_{j=1}^{n}$

$$\beta_1' = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\beta_2' = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$(\beta_1')_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $(\beta_2')_\beta = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

修改

$$\Rightarrow \quad Q = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix}$$
$$\Rightarrow \quad Q^{-1} = \begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix}$$

修改

and
$$\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}}_{\text{Diagonalization of } \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix}$$

Dragonalization is a convenient tool to describe

$$T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$$

Observe: if $B = Q^{-1} A Q \Rightarrow B^m = Q^{-1} A^m Q$ (induction)

and if $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \Rightarrow B^m = \operatorname{diag}(\lambda_1^m, \ldots, \lambda_n^m)$

$$\Rightarrow A^m = Q \operatorname{diag}(\lambda_1^m, \ldots, \lambda_n^m) Q^{-1}.$$

In the elementary example, we have,

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \begin{pmatrix} (-1)^m & 0 \\ 0 & 4^m \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 \\ 3/5 & 3/5 \end{pmatrix}$$

修改

But we still need to find the $\{\gamma_j\}_{j=1}^n$ so that

$$[T]_\gamma^\gamma = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

* Diagonalization

Def^n Given $T \in \mathcal{L}(V,V)$, a vector $\underset{\neq}{\overset{0}{}} v \in V$ is an eigenvector of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. λ is called the corresponding eigenvalue.

Observe: a necessary condition for λ and v is that
$v \in \mathcal{N}(T - \lambda I)$

since $T(v) = \lambda v \Rightarrow T(v) - \lambda v = 0$
$\Rightarrow T(v) - \lambda I(v) = 0$
$\Rightarrow (T - \lambda I)(v) = 0.$

$\therefore$ for any basis β of V, $(v)_\beta \in \mathcal{N}([T-\lambda I]_\beta^\beta)$

since $v \neq 0 \Rightarrow \mathcal{N}([T-\lambda I]_\beta^\beta) \neq \{0\} \Rightarrow [T-\lambda I]_\beta^\beta$ not invertible

$\Rightarrow \det([T - \lambda I_n]_\beta^\beta) = 0$

OR $\underbrace{\det([T]_\beta^\beta - \lambda I_n) = 0}_{\text{eq. to solve for } \lambda.}$

Note: $\det([T]_\beta^\beta - \lambda I_n)$ is a polynomial in λ of degree $\leq n$

$\therefore$ There are at most n eigenvalues,

and for each λ_i, $1 \leq i \leq r \leq n$, we row reduce $[T - \lambda_i I]_\beta^\beta$ to find its $\underbrace{\text{null space}}_{\text{eigenspace for } \lambda_i}$, whose basis elements are corresponding eigenvectors v.

If we can find n independent eigenvectors $v_1, \dots, v_n$
$\Rightarrow$ they form a basis called eigenbasis, and $\gamma = \{v_j\}_{j=1}^{n}$

$$[T]_\gamma^\gamma = \mathrm{diag}(\underbrace{\lambda_1 \cdots \lambda_1}_{m_1} \underbrace{\lambda_2 \cdots \lambda_2}_{m_2} \cdots \underbrace{\lambda_r \cdots \lambda_r}_{m_r})$$

where $m_j =$ nullity of $[T]_\beta^\beta - \lambda_j I_n$

and T (or the matrix $[T]_\beta^\beta$) is called diagonalizable.

eg11 $T: \mathbb{R}^3 \to \mathbb{R}^3$ w/

$$A = [T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \quad ; \quad \beta = \{e_j\}_{j=1}^{3}$$

Find eigenvalues (& eigenvectors) of A

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\therefore$ eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

$$\lambda_1 = 1 \to \mathcal{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \mathrm{Sp}\left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \right\} \; ; \; T(v_1) = v_1$$

$$\lambda_2 = 2 \to \mathcal{N}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Sp}\left\{ \overset{v_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \right\} \; ; \; T(v_2) = 2v_2$$

$$\lambda_3 = 3 \to \mathcal{N}\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ 2x_1 \\ x_1 \end{pmatrix} \right\} = \mathrm{Sp}\left\{ \overset{v_3}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \right\} \; ; \; T(v_3) = 3v_3$$

$$\therefore \text{ w} \cdot \; [T]_\gamma^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix} = \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}$$

修改

OR $$\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix} = \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}$$

and $$\begin{pmatrix}1&1&0\\0&2&2\\0&0&3\end{pmatrix}^{n} = \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2^n&0\\0&0&3^n\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}.$$ //

Of course, things are not always this nice:

we can't always solve $\overbrace{\det(A-\lambda I)}^{\text{characteristic polynomial}}=0$ on $\mathbb{R}$,

and even if we do, we don't always get n distinct roots.

Moreover, even if λ_j is a root of algebraic multiplicity m_j (that is, $\det(A-\lambda I) = (\lambda-\lambda_j)^{m_j}\ldots$) it doesn't always give m_j linearly independent eigenvectors ...

eg.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\det(A - \lambda I_3) = (1-\lambda)^2(3-\lambda) \qquad \left(\begin{matrix} A-\lambda I \text{ is} \\ \text{upper } \triangle \end{matrix}\right)$$

$$\lambda_1 = 3 \rightsquigarrow A - 3I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{nullity} = 1$$

$$\lambda_2 = 1 \rightsquigarrow A - I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{nullity} = 1$$

$\therefore$ only get 2 eigenvectors, can't form eigenbasis $\Rightarrow$ A not diagonalizable.

eg.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I_4) = \det\begin{pmatrix} -\lambda & 0 & 0 & 1 \\ -1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda \det\begin{pmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} - \det\begin{pmatrix} -1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \lambda^4 + 1 \rightsquigarrow \text{no real solution!}$$

$\therefore$ A is not diagonalizable

Facts:

① If $\det(A - \lambda I_n)$ has n distinct roots $\Rightarrow$ A is diagonalizable (HW. problem)

② If $A \in Mat_{n\times n}(\mathbb{R})$ is symmetric ($A = A^T$), then A is diagonalizable

③ If $A \in Mat_{n\times n}(\mathbb{C})$ is "self-adjoint" ($a_{ji} = \overline{a_{ij}}$), then A is diagonalizable

Next semester.

- End of Fall 2014 -