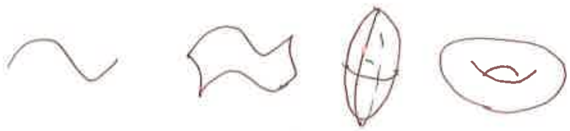
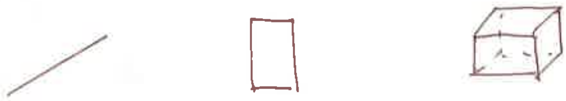
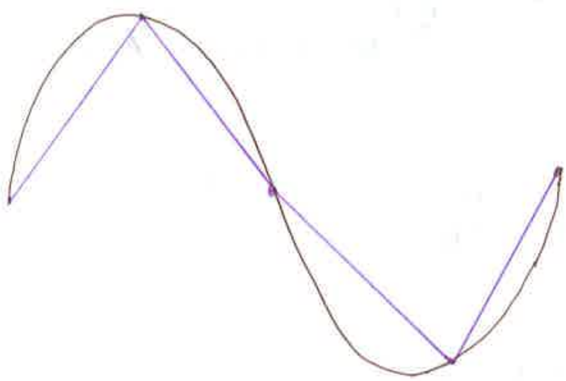
VI. Integration

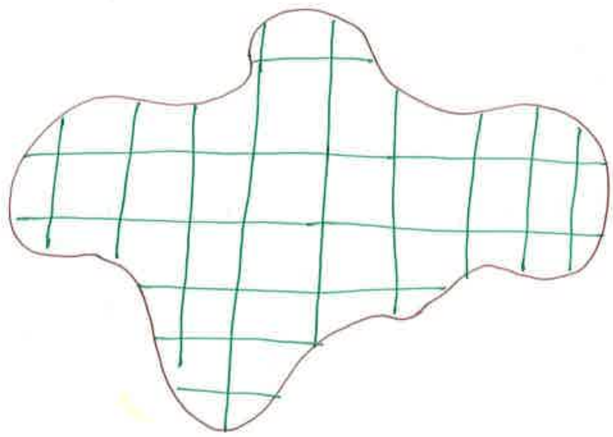
Q: What are the sizes (length, area, volume) of stuff like  ?

We know how to precisely compute sizes of stuff like 

∴ we do what we've done in middle school

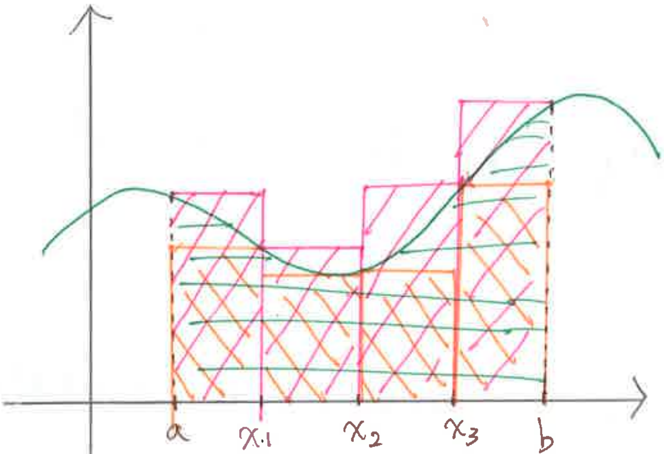
: cut curvey stuff into small pieces of straight stuff with error getting smaller as we cut into more pieces.





cut the objects into "infinitely" many straight pieces (∴ error → 0) and add them up infinitely many times. (We can do so when the curvey objects are described by functions)

Expository Discussion: Area under the curve $y = f(x)$, $a \le x \le b$
(Salas 5.1 - 5.2)



A partition P of $[a,b]$ is $\{x_0, \ldots, x_n\}$.

where
$a = x_0 < x_1 < \ldots < x_n = b$

P' is a refinement of P if $P \subset P'$.

For each i

Let $\left[M_i = \max_{x \in [x_i, x_{i+1}]} f(x) \right]$ consider, rectangles w/ base $[x_i, x_{i+1}]$ height M_i

$\left[m_i = \min_{x \in [x_i, x_{i+1}]} f(x) \right]$ " " " " m_i

$M_i \geq m_i$

M_i's generates area — P upper sum $U_f(P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$

m_i's " " — P lower sum $L_f(P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$

Clearly, for any partition P_n

$L_f(P_n) \le$ Actual Area $\le U_f(P_n)$

Moreover, if $P_{n'}$ is a refinement of P_n $(n' > n)$ (i.e. cut into more pieces)

$L_f(P_n) \le L_f(P_{n'}) \le$ Actual Area $\le U_f(P_{n'}) \le U_f(P_n)$

$n \to \infty$ $\qquad$ $n \to \infty$

# of subdivision

If $\lim_{n\to\infty} U_f(P_n) = \lim_{n\to\infty} L_f(P_n)$, the function f

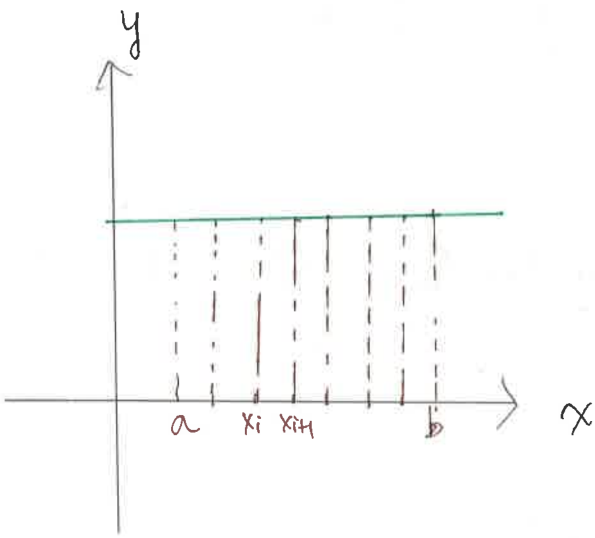
$= \text{Actual Area}$

is said to be integrable on $[a,b]$, and the common limit is denoted $\int_a^b f(x)\,dx$
(actual area)

Fact: A continuous function on $[a,b]$ is always integrable.

Examples,

① $f(x) = K$,



We know

$$\int_a^b f(x) = K(b-a)$$

$M_i = m_i = K$ on all $[x_i, x_{i+1}]$

$$U_f(P_n) = \sum_{i=0}^{n-1} K(x_{i+1} - x_i)$$

$$= K\left[(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \cdots + (x_n - \cancel{x_{n-1}})\right]$$

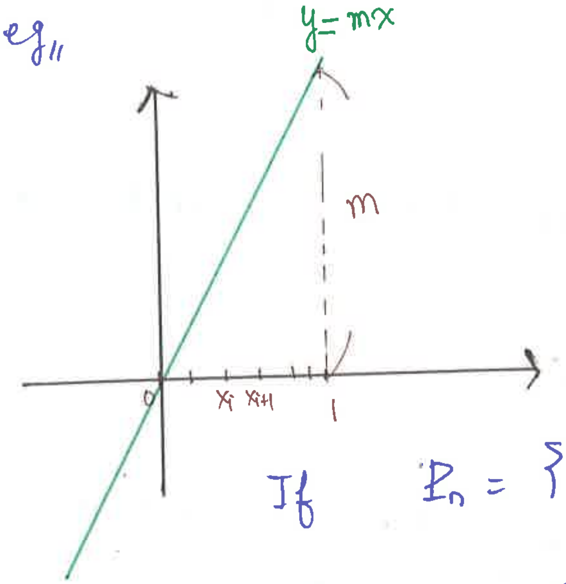
$$= K(x_n - x_0) = K(b-a)$$

Similarly

$$L_f(P_n) = K(b-a)$$

For all n

eg 11



$$\int_0^1 f(x)\,dx = \frac{1}{2}\cdot 1 \cdot m$$

on $[x_i, x_{i+1}]$.

$$M_i = f(x_{i+1}) = m\,x_{i+1}$$

$$m_i = f(x_i) = m\,x_i$$

If $P_n = \{ 0 \le x_0 \le \cdots \le x_{2^n} = 1 \}$

where $x_i = \frac{i}{2^n}$; $x_{i+1} - x_i = \frac{1}{2^n}$

$$U_f(P_n) = \sum_{i=0}^{n-1} m \cdot x_{i+1} \cdot \frac{1}{2^n} = \frac{m}{2^n} \sum_{i=0}^{2^n-1} \frac{i+1}{2^n}$$

$$= \frac{m}{2^{2n}} \frac{(1+2^n)}{2} \cdot 2^n = \frac{1}{2}\cdot\left(1+\frac{1}{2^n}\right) m$$

Similarly, $L_f(P_n) = \frac{1}{2}\left(1 - \frac{1}{2^n}\right) m$

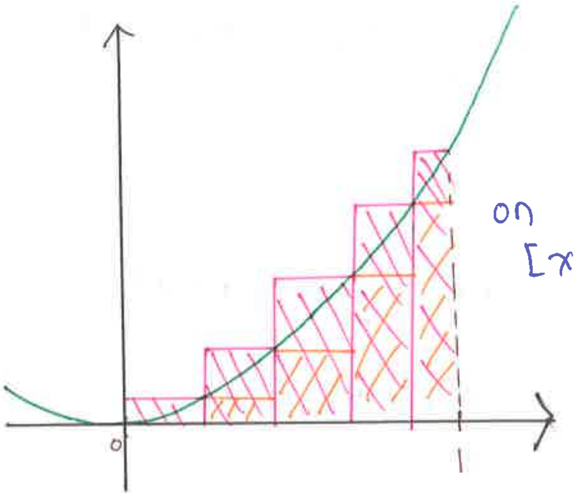
and $\lim_{n\to\infty} L_f(P_n) = \lim_{n\to\infty} U_f(P_n) = \frac{m}{2} = \int_a^b f(x)\,dx$

Infinite/continuum $\quad \int_a^b \quad f(x) \quad dx$

$\updownarrow \qquad$ Height $\qquad$ width

Finite/Discrete $\quad \sum_i \quad f(x^*) \quad \underbrace{(x_{i+1} - x_i)}_{\Delta x_i}$

eg 11 $f(x) = x^2$ $\quad [a,b] = [0,1]$



Divide $[0,1]$ into P_n as in previous example.

on $[x_i, x_{i+1}]$, $\quad M_i = f(x_{i+1}) = \frac{x_{i+1}^2}{2^n}$

$m_i = f(x_i) = \frac{x_i^2}{2}$

$$U_f(P_n) = \sum_{i=0}^{2^n-1} \left(\frac{i+1}{2^n}\right)^2 \cdot \frac{1}{2^n} = \frac{1}{2^{3n}} \sum_{i=0}^{2^n-1} (i+1)^2$$

$$= \frac{1}{2^{3n}} \left[1^2 + 2^2 + \cdots + (2^n)^2\right]$$

Recall $1^2 + 2^2 + \cdots + K^2 = \frac{K(K+1)(2K+1)}{6}$

$$= \frac{1}{2^{3n}} \frac{2^n(2^n+1)(2^{n+1}+1)}{6}$$

$$= \frac{2^{3n+1} + \text{lower}}{3 \cdot 2^{3n+1}} \xrightarrow{n\to\infty} \frac{1}{3}$$

Similarly, $\quad L_f(P_n) = \frac{1}{2^{3n}} \left[1^2 + 2^2 + \cdots + (2^n-1)^2\right]$

$$= \frac{1}{2^{3n}} \frac{(2^n-1) \cdot 2^n \cdot (2^{n+1}-2)}{6}$$

$$= \frac{2^{3n+1} + \text{lower order terms}}{3 \cdot 2^{3n+1}} \xrightarrow{n\to\infty} \frac{1}{3}$$

$\therefore$ Area under the curve $y = x^2$ over $[0,1]$

is $\int_0^1 x^2\,dx = \frac{1}{3}$

- Integral is defined very easily, but computationally very difficult, as opposed to differentiation.

- Examples computed so far can go crazy easily, by change $f(x)$ or endpoints.

$$\int_{\sqrt{2}}^{\sqrt{7}} x^4 \, dx = ?$$

Need more systematic ways to deal with them!

* Fundamental Theorem of Calculus

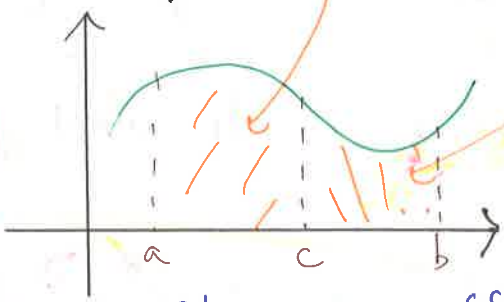
Motivation: find an easier way to compute

$$\int_a^b f(x)\,dx$$

Observation: f integrable on $[a,b]$, $c \in (a,b)$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

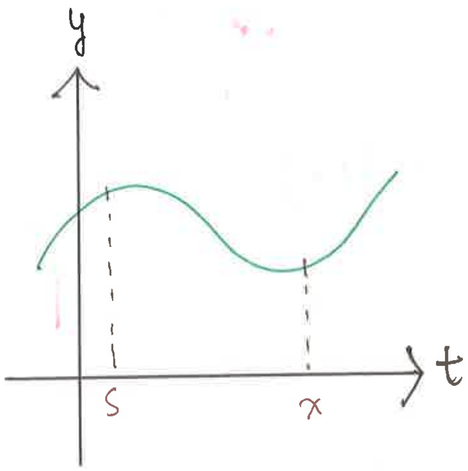
integration is additive.



$$\Rightarrow \quad \int_c^b f(x)\,dx = \int_a^b f(x)\,dx - \int_a^c f(x)\,dx$$

Let $x \to t$, and consider (for fixed s)

$$F(x) := \int_s^x f(t)\,dt$$



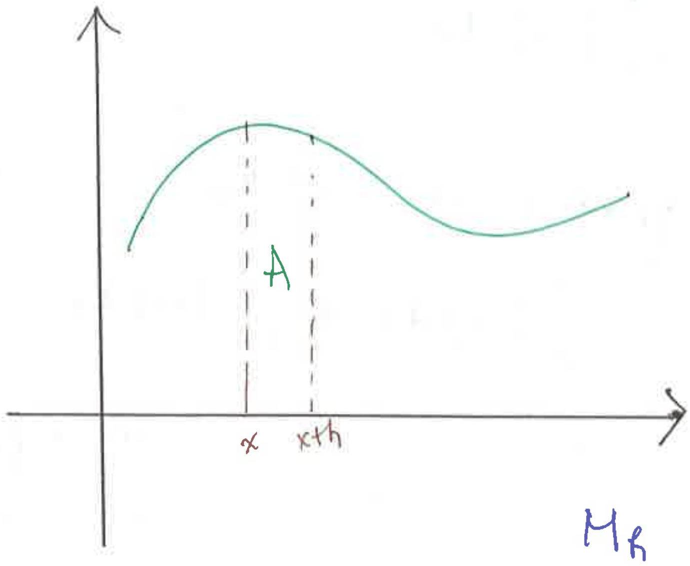
Then clearly, for $s < a < b$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

How to find such F ?

What is the relation between F and f ?

consider $[x, x+h]$ (OR $[x+h, x]$ for $h<0$)



$$A = \int_x^{x+h} f(t)\,dt$$

$$= F(x+h) - F(x)$$

$$M_h = \max_{\substack{x \text{ b/w} \\ x,\, x+h}} f(x)$$

$$m_h = \min_{\substack{x \text{ b/w} \\ x,\, x+h}} f(x)$$

$$M_h \text{ and } m_h \xrightarrow{h \to 0} f(x)$$

$$m_h \cdot h \le F(x+h) - F(x) \le M_h \cdot h$$

$$\begin{array}{ccccc} m_h & \le & \dfrac{F(x+h)-F(x)}{h} & \le & M_h \\ \big\downarrow{\scriptstyle h\to 0} & & & & \big\downarrow{\scriptstyle h\to 0} \\ f(x) & & & & f(x) \end{array}$$

Pinching theorem :

$$\lim_{h \to 0} \frac{F(x+h)-F(x)}{h} = f(x)$$

$$\parallel$$

$$F'(x)$$

If $F(x) = \int_s^x f(t)\,dt \Rightarrow F'(x) = f(x)$

F is called an antiderivative of f

Note: antiderivative is NOT unique:

$(F+K)' = f$ for all constant K

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Thm. (Fundamental Theorem of Calculus)

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b,$$ where $F' = f$ is the antiderivative of f.

Note: constant in F is irrelevant as they cancel out when taking the difference $F(b) - F(a)$.

$\therefore$ may be neglected when computing $\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}}$.

Common antiderivatives:

f	F
x^n	$\frac{x^{n+1}}{n+1} + C$; $n \neq -1$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$

eg.

$$\int_0^1 x^2\,dx = \underbrace{\frac{x^3}{3}\Big|_0^1}_{\substack{F(1) - F(0) \\ \text{w/ } F = \frac{x^3}{3}}} = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$\int_2^4 x^4\,dx = \frac{x^5}{5}\Big|_2^4 = \frac{\cdot 6 - 32}{5} = \frac{224}{5}$$

Basic Properties:

(1) Integration is linear:

$$\int_a^b (\alpha_1 f_1 + \cdots + \alpha_n f_n)\,dx = \alpha_1 \int_a^b f_1(x) + \cdots + \alpha_n \int_a^b f_n(x)\,dx$$

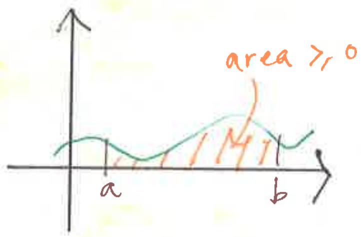
since, if F_i = antiderivative of f_i. $F_i' = f_i$

$$(\alpha_1 F_1 + \cdots + \alpha_n F_n)' = \alpha_1 F_1' + \cdots + \alpha_n F_n'$$
$$= \alpha_1 f_1 + \cdots + \alpha_n f_n$$

$$\therefore \int_a^b (\alpha_1 f_1 + \cdots + \alpha_n f_n)\,dx = \alpha_1 F_1\Big|_a^b + \cdots + \alpha_n F_n\Big|_a^b$$
$= RHS.$

(2) Integration is monotonic on continuous functions.

f cont. on $[a,b]$ and $f \geq 0$ $\Rightarrow$ $\int_a^b f(x)\,dx \geq 0$



$\quad$ " " $\quad f > 0 \Rightarrow \int_a^b f(x)\,dx > 0$

This implies: (2a) if $f \geq g \Rightarrow \int_a^b f\,dx \geq \int_a^b g\,dx$

$f > g \Rightarrow \int_a^b f\,dx > \int_a^b g\,dx$

(2b) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

Larger (smaller) Functions yield larger (smaller) integrals.

since $-|f(x)| \leq f(x) \leq |f(x)|$

$$\overset{2a}{\Rightarrow} -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$(\ -b \leq a \leq b \rightsquigarrow |a| \leq b\ )$

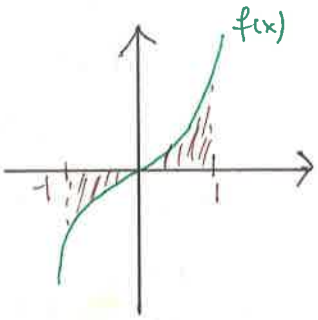
(2c) if $m \leq f(x) \leq M$ on $[a,b]$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(3) Signs on Integrals

(3a) Areas generated by curves below x-axis ($f \le 0$) are considered negative. (since we take $f(x)$ as the height of rectangles, not $|f(x)|$)

eg.



$$\int_{-1}^{1} f(x)\,dx = 0$$

(3b) Switching the limits of integration alters the sign:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\begin{aligned} 0 &= \int_a^a f(x)\,dx \\ &= \int_a^b f(x)\,dx + \int_b^a f(x)\,dx \end{aligned}$$

(3b) f is odd $\Rightarrow$ $\int_{-a}^{a} f(x)\,dx = 0$

is even $\Rightarrow$ $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$\int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-x)\,dx \overset{3b}{=} \int_0^a f(-x)\,dx$$

$$= \begin{cases} \int_0^a f(x)\,dx, & f \text{ even} \\ -\int_0^a f(x)\,dx, & f \text{ odd} \end{cases}$$

(4) $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)$

$$G(x) = \int_{g(x)}^{h(x)} f(t)\,dt = \int_a^{h(x)} f(t)\,dt + \int_{g(x)}^{a} f(t)\,dt$$

$$= \underbrace{\int_a^{h(x)} f(t)\,dt}_{F_1(x)} - \underbrace{\int_a^{g(x)} f(t)\,dt}_{F_2(x)}$$

$F_1(x) = G_1 \circ h\ (x)$ where $G_1(u) = \int_a^u f(t)\,dt$.

$\therefore\ F_1'(x) = G_1'(h(x)) \cdot h'(x)$ $\Rightarrow \frac{dG_1}{du} = f(u)$

$= f(h(x)) \cdot h'(x)$

Similarly, $F_2'(x) = f(g(x)) \cdot g'(x)$.

eg.

$$\int_0^3 \left(x^2 + \cos x - \frac{3}{\sqrt{x}}\right) dx$$

$$= \int_0^3 x^2\,dx + \int_0^3 \cos x\,dx - 3\int_0^3 x^{-\frac{1}{2}}\,dx$$

$$= \frac{x^3}{3}\Big|_0^3 + \sin x\Big|_0^3 - 3\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^3$$

$$= (9-0) + (\sin 3 - \sin 0) - 6(\sqrt{3} - \sqrt{0})$$

$$= 9 + \sin 3 - 6\sqrt{3} \quad //$$

eg.

$$\int_{-2}^{2} |x|\,dx \qquad |x| = \begin{cases} -x, & -2 \le x \le 0 \\ x, & 0 \le x \le 2 \end{cases}$$

$$= \int_{-2}^{0} -x\,dx + \int_0^2 x\,dx$$

$$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^2 = \left(-\frac{0}{2} + \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{0}{2}\right) = 4$$

But notice $|-x| = |x| \Rightarrow |x|$ is even

$$\therefore \int_{-2}^{2} |x|\,dx = 2\int_0^2 x\,dx = 4.$$

eg.

$$\frac{d}{dx}\int_{\underset{g(x)}{x^2}}^{\overset{h(x)}{2x}} \overset{f(t)}{\frac{1}{1+t^2}}\,dt$$

$$= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

$$= \frac{1}{1+(2x)^2}\cdot 2 - \frac{1}{1+x^4}\cdot 2x$$

eg: Find $H'(2)$ given

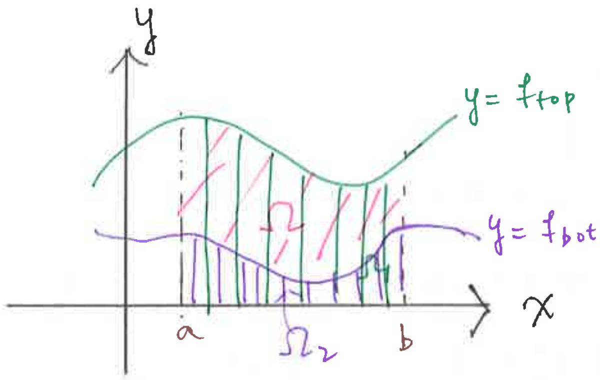
$f(t)$

$$H(x) = \int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}}\, dt$$

$$H'(x) = \frac{x}{1+\sqrt{x^3-4}}(3x^2) - \frac{2x}{1+\sqrt{2x}}$$

$$H'(2) = \frac{2}{1+\sqrt{4}} \cdot 12 - \frac{4}{1+\sqrt{4}} = 8 - \frac{4}{3} = \frac{20}{3}.$$

* Area Between Curves.

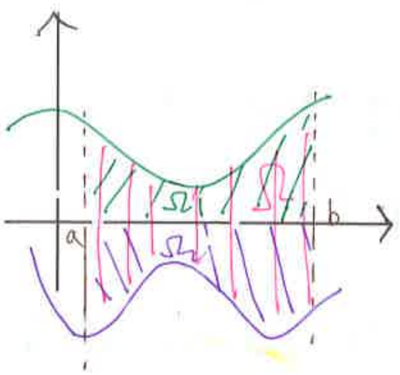


compute the absolute area between two curves $y = f_{top}$ and $y = f_{bot}$

always +, even below x-axis

$$\text{Area } \Omega = \text{Area } \Omega_1 - \text{Area } \Omega_2$$
$$= \int_a^b f_{top}\, dx - \int_a^b f_{bot}\, dx$$
$$= \int_a^b (f_{top} - f_{bot})\, dx$$

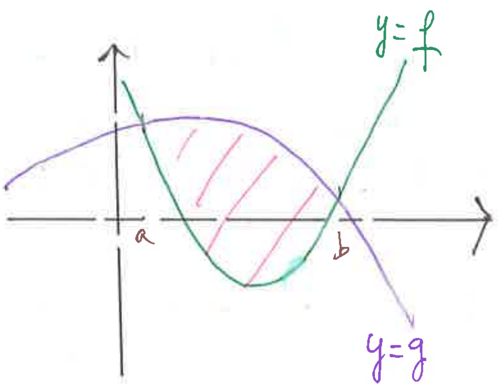
This expression holds when functions are negative



$$\text{Area } \Omega = \text{Area } \Omega_1 + \text{Area } \Omega_2$$
$$= \int_a^b f_{top}\, dx - \int_a^b f_{bot}\, dx$$
$$= \int_a^b (f_{top} - f_{bot})\, dx$$

Area Bounded by two curves: $y = f(x)$ and $y = g(x)$

a, b = points where $f = g$



Sketch the curves to check which is f_{top} and f_{bot}

$$\Rightarrow \Omega = \int_a^b (f_{top} - f_{bot})\, dx$$

eg 11 Compute area bounded by

$$y = \underbrace{5 - x^2}_{f}, \quad y = \underbrace{3 - x}_{g}$$

$5 - x^2 = 3 - x \Rightarrow x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$, $x = \overset{a}{-1}, \overset{b}{2}$

$f(0) = 5$, $g(0) = 3$

$\therefore f = f_{top}$ $g = f_{bot}$

$$\begin{aligned} \text{Area} &= \int_{-1}^{2} (5 - x^2 - 3 + x)\,dx \\ &= \int_{-1}^{2} (-x^2 + x + 2)\,dx \\ &= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{-1}^{2} \\ &= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) \\ &= \frac{9}{2} \end{aligned}$$

Area bounded by curves intersecting multiple times

f_{top}, f_{bot} may be switched.

Find the area bounded by $y = 4x$ and $y = x^3$

$x^3 = 4x \Rightarrow x = -2, 0, 2$

$-2 \le x \le 0$

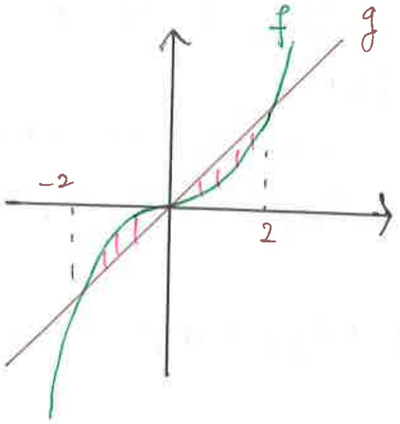
$f_{top} = x^3$,

$f_{bot} = 4x$

$0 \le x \le 2$

$f_{top} = 4x$

$f_{bot} = x^3$



$$\text{Area} = \int_{-2}^{0} (x^3 - 4x)\,dx + \int_{0}^{2} (4x - x^3)\,dx$$

$$= 8$$

11/28 Ends

* More on Antiderivatives

Recall, antiderivative F of f is a function s.t. $F' = f$.

without limit   any constant

We also write $\int f\,dx = F + C$

called the "indefinite integral of f".

($\because$ $\int_a^b f\,dx \in \mathbb{R}$ is called definite integral of f over $[a,b]$)

eg// $\int (t-a)(t-b)\,dt = \int [t^2 - (a+b)t + ab]\,dt$

$$= \frac{t^3}{3} - (a+b)\frac{t^2}{2} + ab\,t + C.$$

The constant C is determined by one particular point $(x_0, f(x_0))$ on the graph

eg// Find F s.t. $F' = x^2+1$ and $F(0) = 4$

$$F = \int (x^2+1)\,dx = \frac{x^3}{3} + x + C$$

$$F(0) = C = 4 \Rightarrow F(x) = \frac{x^3}{3} + x + 4$$

Finding f given $f^{(n)}$:

$$f^{(n-1)} = \int f^{(n)} + c_1$$
$$f^{(n-2)} = \int f^{(n-1)} + c_2 = \iint f^{(n)} + c_1 x + c_2$$
$$\vdots$$
$$f = \underbrace{\iint \cdots \int}_{n} f^{(n)} + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

$\therefore$ Need n particular conditions

$(x_1, f^{(i_1)}(x_1)), \ldots, (x_n, f^{(i_n)}(x_n))$ to determine

$c_1, \ldots, c_n$

eg. Find f given $f'' = \cos x$. $f'(0)=1$, $f(0)=2$

$$f' = \int f'' dx = \sin x + c_1$$

$$f'(0) = c_1 = 1 \Rightarrow f' = \sin x + 1$$

$$f = \int f' dx = -\cos x + x + c_2$$

$$f(0) = -1 + c_2 = 2 \Rightarrow c_2 = 3$$

$$\therefore f(x) = -\cos x + x + 3$$

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* Particle Motion

· A particle moves along straight line with position described by $x(t)$ ← time on the real line

Rate of change of position, or "velocity" is

$$v(t) = x'(t) = \frac{dx}{dt}$$

$v(t) > 0 \iff x(t)$ increasing
< 0 — decreasing

$\iff$ particle moving right
" " left

Rate of change of velocity, or "acceleration" is

$$a(t) = v'(t) = x''(t)$$
$$\left(\frac{dv}{dt}\right) \qquad \left(\frac{d^2x}{dt^2}\right)$$

$a(t) > 0 \iff v(t)$ increasing
< 0 " decreasing

Speed $Sp(t) = |v(t)|$ = how fast the particle moves regardless of direction

$$\frac{d}{dt} Sp(t) = \begin{cases} \frac{dv}{dt}; & v \geq 0 \\ -\frac{dv}{dt}; & v < 0 \end{cases}$$

v	$\frac{dv}{dt}$	$\frac{d}{dt}Sp(t)$
+	+	+
+	−	−
−	+	−
−	−	+

particle speeds up

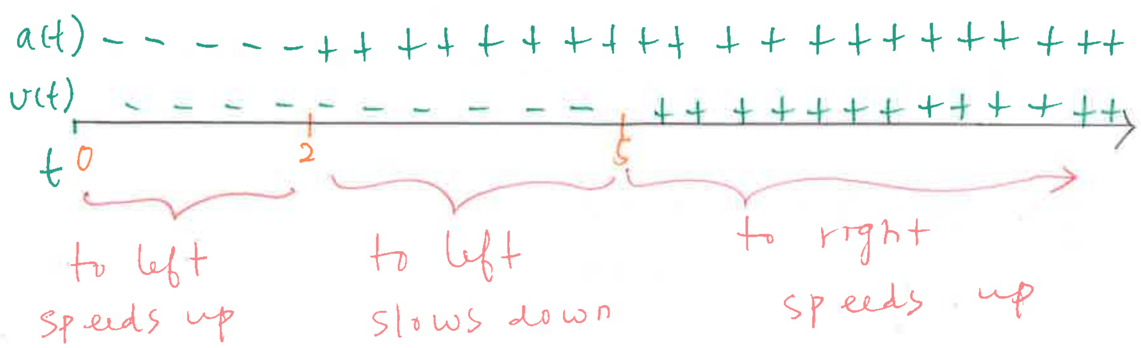
particle slows down

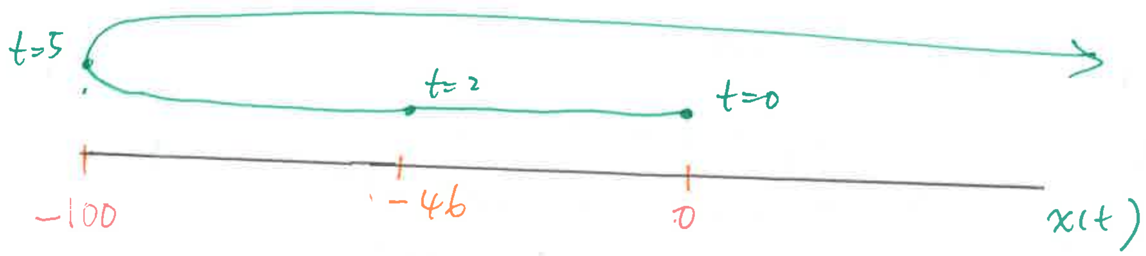
eg.. Describe the motion $x(t) = t^3 - 6t^2 - 15t$

$$v(t) = x'(t) = 3t^2 - 12t - 15$$
$$= 3(t+1)(t-5)$$

Speed $|v(t)| = \begin{cases} 3(t+1)(t-5) & ; t \geq 5 \\ 3(t+1)(5-t) & ; 0 \leq t \leq 5 \end{cases}$

$$a(t) = 6t - 12 = 6(t-2)$$





Working backward,

Given $a(t)$

$v'(t) = a(t) \Rightarrow \quad v(t) = \int a(t')dt' + c_1$

$x(t) = x'(t) \Rightarrow \quad x(t) = \int v(t')dt' + c_2$

c_1, c_2 are determined by 2 values of $a(t)$ (or $v(t)$) at some point(s)

eg// $a(t) = a$, constant

$\Rightarrow \quad v(t) = \int a\,dt = at + c_1$

$\Rightarrow \quad x(t) = \int v\,dt = \frac{1}{2}at^2 + c_1 t + c_2$

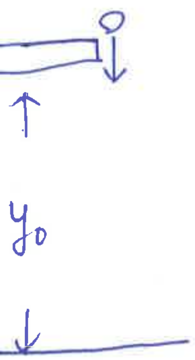
if $v(0) = v_0$, initial velocity $\Rightarrow c_1 = v_0$

$x(0) = x_0$, intial position $\Rightarrow c_2 = x_0$

$x(t) = \frac{1}{2}at^2 + v_0 t + x_0.$

Free fall, $x(t) \rightsquigarrow y(t)$

$a = -g \quad (-4.9\ m/s^2)$

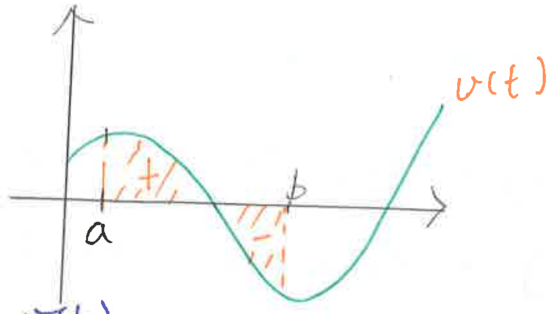


$y(t) = -\frac{g}{2}t^2 + v_0 t + y_0.$

Displacement between
$a \le t \le b$

$$x(b) - x(a) = \int_a^b v(t)\,dt$$

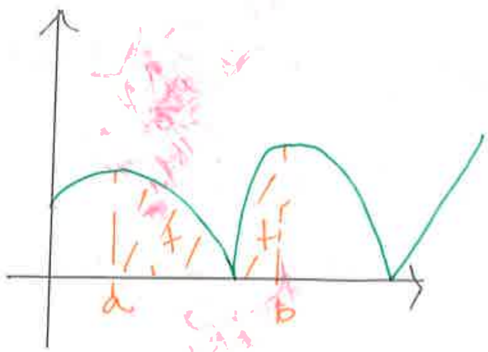
= signed area under / above $y = v(t)$



Distance Traveled
Between $a \le t \le b$

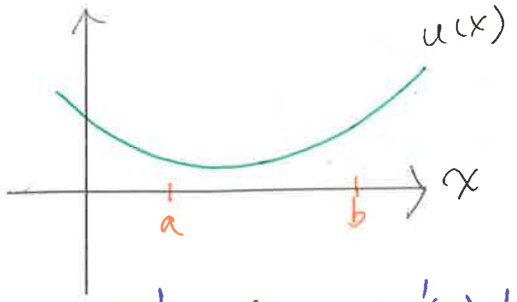
$$= \int_a^b |v(t)|\,dt$$

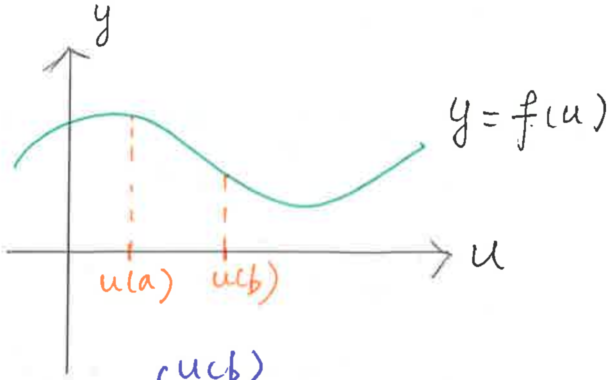
$$= \int_a^b Sp.(t)\,dt$$



Rmk: think of $\int_a^b f\,dx$ as sum of areas infinitely many small rectangles of height $f(x)$ and width dx.

* u - substitution (Change of Variable)





differential → $du = u'(x)\,dx$

$\left(\text{OR } dx = \dfrac{1}{u'(x)}\,du\right)$

$$\int_{u(a)}^{u(b)} f(u)\,du = \int_a^b f(u(x))\cdot u'(x)\,dx$$

Application :

eg//

$$\int (x^2-1)^4 x\,dx$$

$$= \int u^4 \cdot \cancel{x} \cdot \frac{du}{2\cancel{x}}$$

$u = x^2 - 1$

$u' = 2x$

$$= \frac{1}{2}\int u^4\,du = \frac{1}{10}u^5 + C = \frac{1}{10}(x^2-1)^5 + C$$

eg//

$$\int 2x^3 \sec^2(x^4+1)\,dx$$

$$= \int 2\cancel{x^3} \sec^2 u \cdot \frac{du}{4\cancel{x^3}}$$

$u = x^4 + 1$

$u' = 4x^3$

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan(x^4+1) + C.$$

eg 11

$$\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\,dx \qquad u = x^2+1$$

$$u' = 2x$$

$$= \int^4 x^{\cancel{5}4}\sqrt{u}\,\frac{du}{\cancel{2x}} \qquad x^4 = (u-1)^2$$

$$= \frac{1}{2}\int_1^4 (u-1)^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int_1^4 (u^2-2u+1)u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_1^4 (u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}})\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}}-\frac{4}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^4 = \frac{848}{105} \quad //$$

* Mean-Value Theorem for Integrals

Thm. $F(x) = \int_a^x f\,dx$ is differentiable on (a,b)

if f cont. on $[a,b]$

$F(b) = \int_a^b f(x)\,dx$, $F(a) = 0$. By Mean-value thm.

on $F(x)$, $\exists\, c \in (a,b)$ s.t.

$$\frac{F(b) - F(a)}{b - a} = F'(c) = f(c)$$

$$\frac{\int_a^b f\,dx}{b-a}$$

We have

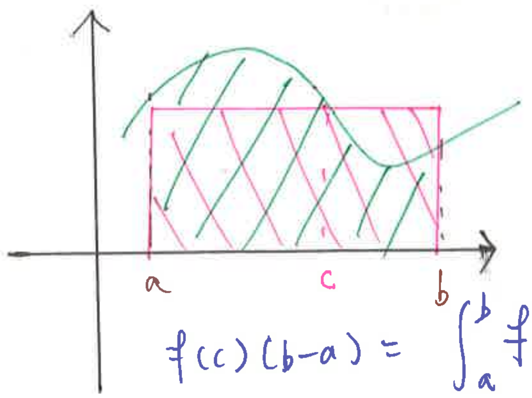
Thm. (Mean-Value Thm. for integral)

f cont. on $[a,b]$. there is $c \in (a,b)$ s.t.

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

average value of f on $[a,b]$

$$= \frac{\int_a^b f(x)\,dx}{b-a}$$



$f(c)(b-a) = \int_a^b f(x)\,dx$

eg. $f(x) = \frac{1}{x^2}$, $x \in [1,4]$

$$f_{ave} = \frac{\int_1^4 \frac{1}{x^2}\,dx}{4-1} = \frac{\left.\frac{-1}{x}\right|_1^4}{3} = \frac{1}{4}$$

which occurs at $x = 2$. //