X. Invertibility & Determinant

Know: $A \in Mat_{n\times n}$ is invertible $\Leftrightarrow rk(A) = n$

$\Leftrightarrow N(L_A) = \{0\}$.

$\Leftrightarrow rref(A) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Any other systematic way to check it?

$n=1$ $\quad A = (a_{11})$, $\quad x \in \mathbb{R}^1$

$x \in \mathbb{R}^1$ $\quad \underset{\text{the transformation}}{\underbrace{A x}} = a_{11} x$ is invertible $\Leftrightarrow a_{11} \neq 0$

$(A^{-1} = (a_{11}^{-1}))$

$n=2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

case 1: $a = 0$

A invertible $\Leftrightarrow b, c \neq 0$

$\Leftrightarrow bc \neq 0$

$\Leftrightarrow ad - bc \neq 0$

case 2: $a \neq 0$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{E^{a\cdot 2}} \begin{pmatrix} a & b \\ ac & ad \end{pmatrix} \xrightarrow{E^{-c\cdot 1+2}} \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}$$

$\therefore rk(A) = 2 \Leftrightarrow ad - bc \neq 0$

$\therefore A$ invertible $\Leftrightarrow \underbrace{ad - bc}_{\det A} \neq 0$

Some obvious Facts:

① det. is linear on each row/column.

$$\det(u+v,\ w) = \det(u\ w) + \det(v\ w) \quad \forall\, u, v, w \in \mathbb{R}^2$$

$$\det(\lambda u\ w) = \lambda \det(u \cdot w) \quad \forall\, \lambda \in \mathbb{R},\ \lambda \neq 0$$

② Switching two rows (columns) switches the sign of the determinant.

$$\det(w\ u) = -\det(u\ w)$$

$$\Rightarrow \det(w\ w) = -\det(w\ w) \Rightarrow \det(w\ w) = 0.$$

$$\Rightarrow \det(u+\lambda w\ \ w) = \det(u\ w) + \lambda \det(w\ w)$$

③ ①, ② hold on rows as well.

$$\det\begin{pmatrix} u+v \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + \det\begin{pmatrix} v \\ w \end{pmatrix} \quad \forall\, u^T, v^T, w^T \in \mathbb{R}^2$$

$$\det\begin{pmatrix} \lambda u \\ v \end{pmatrix} = \lambda \det\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\det\begin{pmatrix} u+\lambda v \\ v \end{pmatrix} = \det\begin{pmatrix} u \\ v \end{pmatrix}$$

∴ Effect of elementary matrices on determinants:

∴ $\det(AE^{kl}) = -\det(A) = \det(E^{kl}A)$

$\lambda \neq 0$ $\quad \det(AE^{\lambda j}) = \det(E^{\lambda \cdot i} \cdot A) = \lambda \det(A)$

$\det(AE^{\lambda k+l}) = \det(E^{\lambda q+r} \cdot A) = \det(A)$.

H.W. Prove these facts for $A \in Mat_{n\times n}$

* Determinants of $n\times n$ Matrices

$A = (a_{ij}) \in \mathrm{Mat}_{n\times n}$.

Def. (minor)
The ijTH minor of A is $\underset{\text{the matrix}}{\widetilde{A^{ij}}}$, formed deleting iTH row and jTH column from A.

Def. (Formula for Determinant)

$n=1 \Rightarrow \quad \det A = a_{11}$

$n>1 \Rightarrow \quad \det A = \sum_{j=1}^{n} (-1)^{1+j} \det(\widetilde{A^{1j}})$

eg.
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\det(\widetilde{A^{11}}) - b(\det\widetilde{A^{12}}) = ad-bc$$

eg.
$$\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} = 1\cdot\det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} - 3\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + 3\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix}$$
$$= 40.$$

Routine (but Tedious) to check:

Thm 1
det is linear on each row / column.

$$\det\begin{pmatrix} R_1 \\ \vdots \\ \lambda R_i + R_i' \\ \vdots \\ R_n \end{pmatrix} = \lambda\det\begin{pmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1 \\ \vdots \\ R_i' \\ \vdots \\ R_n \end{pmatrix}$$

$$\det(C_1 \cdots \lambda C_j + C_j' \cdots C_n) = \lambda\det(C_1 \cdots C_n) + \det(C_1 \cdots C_j' \cdots C_n)$$

- induction on n -

Thm 2 Cofactoring may takes place at any row.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \det(\widetilde{A^{ij}})$$

– tedious, but easy proof –

Thm 3 If $A \in Mat_{n\times n}$ has two identical rows
$\det A = 0$.

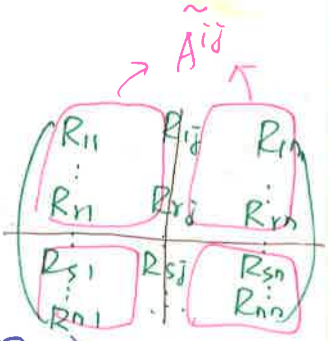
Pf: induction on n:

$n=2$ $\quad \det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = 0$ ✓

suppose the statement is true for some $n \geq 3$

let $A \in Mat_{(n+1)\times(n+1)}$, with $R_r = R_s$

$n \geq 3 \Rightarrow n+1 > 3$, take $i \notin \{r, s\}$.



$$\det A = \sum_{j=1}^{n+1} (-1)^{i+j} \det(\widetilde{A^{ij}})$$

since $i \notin \{r,s\}$ $\quad \widetilde{R_{rj}} = (R_{r1} \ldots R_{r,j-1} \; R_{r,j+1} \ldots R_n)$

$\widetilde{R_{sj}} = (R_{s1} \ldots R_{s,j-1} \; R_{s,j+1} \ldots R_n)$

are both rows of $\widetilde{A^{ij}}$

since $R_r = R_s \Rightarrow \widetilde{R_r} = \widetilde{R_s}$ $\therefore \forall j,\ \widetilde{A^{ij}}$ has two identical rows.

$\widetilde{A^{ij}} \in Mat_{n\times n}$, $\therefore$ induction hypothesis

$\Rightarrow \det \widetilde{A^{ij}} = 0 \quad \forall j$

$\Rightarrow \det A = 0$

QED.

Effects of Elementary Operations on Determinants.

Thm 4. Switching two rows (columns) switches the sign of det.

ie. $\det(E^{kl}A) = \det(A \cdot E^{kl}) = -\det A$.

Pf:

$$A = \begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_l \\ \vdots \\ R_n \end{pmatrix} \quad ; \quad E^{kl}A = \begin{pmatrix} R_1 \\ \vdots \\ R_l \\ \vdots \\ R_k \\ \vdots \\ R_n \end{pmatrix}$$

$$0 = \det\begin{pmatrix} R_1 \\ \vdots \\ R_k + R_l \\ \vdots \\ R_k + R_l \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_k + R_l \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1 \\ \vdots \\ R_l \\ \vdots \\ R_k + R_l \\ \vdots \\ R_n \end{pmatrix}$$

$$= \underbrace{\det\begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_k \\ \vdots \\ R_n \end{pmatrix}}_{0} + \det\begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_l \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1 \\ \vdots \\ R_l \\ \vdots \\ R_k \\ \vdots \\ R_n \end{pmatrix} + \underbrace{\det\begin{pmatrix} R_1 \\ \vdots \\ R_l \\ \vdots \\ R_l \\ \vdots \\ R_n \end{pmatrix}}_{0}$$

$$= \det(A) + \det(E^{kl}A) \Rightarrow \det(E^{kl}A) = -\det(A)$$

Similarly for $\det(AE^{kl})$. QED.

Thm 5: Multiplying a row (column) by $\lambda \in \mathbb{R}$, multiplies the determinant by λ.

(linearity of det on each row/column.)

Thm 6 Adding λR_k into R_l doesn't affect det.

$$\det(E^{\lambda k+l}A) = \det(A E^{\lambda k+l}) = \det A$$

pf.,

pf//

$$A = \begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_l \\ \vdots \\ R_n \end{pmatrix}$$

$$E^{\lambda k+l}A = \begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ \lambda R_k + R_l \\ \vdots \\ R_n \end{pmatrix}$$

$$\det(E^{\lambda k+l}\cdot A) = \underbrace{\lambda \det\begin{pmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_k \\ \vdots \\ R_n \end{pmatrix}}_{0} + \det\begin{pmatrix} R_1 \\ \vdots \\ R_l \\ \vdots \\ R_n \end{pmatrix} = \det A.$$

Similarly for $\det(A\cdot E^{\lambda k+l})$. QED//

$\therefore$ It may saves time to apply row/column operations before computing determinant:

eg.,, $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{E^{-2\cdot 1+2}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{E^{-1\cdot 2+3}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

$\xrightarrow{E^{-1\cdot 2+1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{E^{-\frac{1}{3}\cdot 3+2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{E^{\frac{1}{3}\cdot 3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$1 = \det I_3 = \det\Big(\underbrace{E^{\frac{1}{3}\cdot 3}}_{\frac{1}{3}}\ \underbrace{E^{-\frac{1}{3}\cdot 3+2}\cdot E^{-1\cdot 2+1}\cdot E^{-1\cdot 2+3}\cdot E^{-2\cdot 1+2}}_{\text{no effect on det } A} A\Big)$$

$$= \tfrac{1}{3}\det A \Rightarrow \det A = 3$$

* Determinants and Invertibilities.

Recall: $E^{kl} I_n$ switches k_{TH} & l_{TH} row of I_n $\Rightarrow$ $\det E^{kl} = -\det(I_n)$
$= 1$

$E^{\lambda k} I_n$ multiplies k_{TH} row of I_n
$\Rightarrow \det(E^{\lambda k} I_n) = \lambda \det(I_n)$
$= \lambda$

$E^{\lambda k + l} \cdot I_n$ adds k_{TH} row of I_n to its l_{TH} row
$\Rightarrow \det(E^{\lambda k+l}) = \det(E^{\lambda k+l} \cdot I_n)$
$= \det I_n = 1.$

Thm// $\det(AB) = \det A \cdot \det B$

Pf We've proved the case when A is elementary matrix.:

$A = E^{kl}$ $\quad \det(E^{kl} B) = -\det B$
$= \det(E^{kl}) \det B$

$A = E^{\lambda k}$ $\quad \det(E^{\lambda k} B) = \lambda \det B$
$= \det(E^{\lambda k}) \det B$

$A = E^{\lambda k + l} \cdot B$ $\quad \det(E^{\lambda k + l} B) = \det B$
$= \det(E^{\lambda k + l}) \det B.$

If A is not invertible, rank $A < n$

$\exists$ $E_1, \ldots, E_p$ elementary matrices w/ $\det(E_i) \neq 0$ so that

$$E_1 \cdots E_p A = \left(\begin{array}{c} * \\ \hline 0 \ 0 \cdots 0 \end{array}\right)$$

$$\Rightarrow E_1 \cdots E_p A B = \left(\begin{array}{c} * \\ \hline 0 \cdots 0 \end{array}\right)$$

$$0 = \det(E_1 \cdots E_p A B)$$
$$= \det(E_1 \cdots E_p) \det(AB)$$
$$= \underbrace{\det(E_1) \cdots \det(E_p)}_{\neq 0} \det(AB)$$

$$0 = \det(E_1 \cdots E_p A)$$
$$= \det(E_1 \cdots E_p) \det A$$
$$= \underbrace{\det E_1 \cdots \det E_p}_{\neq 0} \det A$$

$\leadsto$

$$\therefore \quad 0 = \det(AB) = \det A \det B \quad \checkmark$$

For A with rank n.

$A = E_1 \cdots E_p$, E_i elementary matrices and $\det E_i \neq 0$

$$\det(AB) = \det(E_p \cdots E_1 B)$$
$$= \det E_p \det(E_{p-1} \cdots E_1 B)$$
$$= \det E_p \det E_{p-2} \det(E_{p-3} \cdots E_1 B)$$
$$\vdots$$
$$= \det E_p \det E_{p-2} \cdots \det E_1 \det B$$
$$= \det(E_p \cdots E_1) \det B = \det A \det B. \quad \text{QED}$$

Thm 8. A is invertible $\Leftrightarrow$ $\det A \neq 0$.

pf. $\Rightarrow$ A invertible. $A A^{-1} = I_n$

$$\det(A A^{-1}) = 1$$
$$\det(AA^{-1}) = \det A \cdot \det A^{-1}$$
$$\therefore \det A,\ \det A^{-1} \neq 0$$

(also, $\det A^{-1} = \dfrac{1}{\det A}$)

$\Leftarrow$ If A not invertible, $\exists\ E_1, \ldots, E_p$ w/ $\det E_i \neq 0$

s.t.

$$E_p \cdots E_1 A = \left(\begin{array}{ccc} & \ast & \\ \hline 0 & \cdots & 0 \end{array}\right)$$

$$\det(E_p \cdots E_1 A) = 0$$
$$\det(E_p \cdots E_1 A) = \underbrace{\det(E_p) \cdots \det(E_1)}_{\neq 0} \det A \rightsquigarrow \det A = 0$$

QED

For a system of n equations with n unknowns,

$$Ax = b \quad , \quad A \in Mat_{n\times n}$$

We see that the system has a unique solution

$\iff \det A \neq 0.$

If $\det A \neq 0$, there is a formula to solve each x_k in $x = (x_1, \dots, x_n)^T$.

Thm 9 (Cramer's Rules) { For $Ax = b$ with $\det A \neq 0$

$$x_k = \frac{\det M_k}{\det A}$$

where M_k is formed by replacing k_{TH} column of A by x.

Pf: Let $X_k \in Mat_{n\times n}$ be formed by replacing k_{TH} column of I_n by x.

$$X_k = \begin{array}{c} \begin{array}{ccc} & k & \end{array} \\ \begin{pmatrix} 1 & x_1 & \\ \ddots & \vdots & 0 \\ 0 & x_k & \\ & \vdots & \ddots \\ & x_n & 1 \end{pmatrix} \end{array} = (e_1 \; e_2 \cdots e_{k-1} \; x \; e_{k+1} \cdots e_n)$$

$$A X_k = (Ae_1 \; Ae_2 \cdots Ax \cdots Ae_n)$$

if x solves $Ax = b$

$$= (Ae_1 \; Ae_2 \cdots b \cdots Ae_n)$$

recall $Ae_j = j_{TH}$ row of A

$$= M_k$$

CONT.,

also $\det X_k = \sum_{j=1}^{n} (-1)^{k+j} (X_k)_{kj}$

$$= (-1)^{k+k} x_k \det I_{n-1} = x_k .$$

$$\therefore \quad \left.\begin{array}{l} \det(A X_k) = \det(M_k) \\ \qquad \| \\ \det A \det X_k = x_k \det A \end{array}\right\} \quad x_k = \frac{\det M_k}{\det A}.$$

QED.,

eg.,

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ x_1 \qquad + x_3 = 3 \\ x_1 + x_2 - x_3 = 1 \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} ; \quad M_1 = \begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} ; \quad M_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix} ;$$

$$M_3 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$\det A = 6, \quad \det M_1 = 15, \quad \det M_2 = -6, \quad \det M_3 = 3$

$\rightsquigarrow \quad x_1 = \frac{5}{2}, \quad x_2 = -1 \quad x_3 = \frac{1}{2}$

Thm. $\forall A \in Mat_{n\times n}$, $\det A^t = \det A$

Pf. The statement is true for elementary matrices:

since E^{kl}, $E^{\lambda k}$ are symmetric matrices,

$$(E^{kl})^T = E^{kl}, \qquad (E^{\lambda k})^T = E^{\lambda k}$$

$$E^{\lambda k+l} = \begin{array}{c} \\ \\ k\to \\ \\ l\to \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} & & k & & l & \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & & \vdots & & & \vdots \\ \vdots & \vdots & \ddots & 1 & & \lambda & & & 0 \\ & & & \vdots & \ddots & & & & \vdots \\ 0 & 0 & & 0 & & 1 & & & \\ 0 & 0 & & 0 & \cdots & 0 & & \cdots & 1 \end{pmatrix} \end{array}$$

$$(E^{\lambda k+l})^T = \begin{array}{c} \\ \\ k\to \\ \\ l\to \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} & & k & & l & \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & & & 0 & & \vdots & & \vdots \\ \vdots & & & 1 & & 0 & & \vdots \\ \vdots & & & \lambda & & 1 & & \vdots \\ 0 & & & 0 & & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\det(E^{\lambda k+l})^T \underset{\substack{\uparrow \\ \text{cofactor on } k\text{TH row}}}{=} (-1)^{l+k}\lambda \underbrace{\det\begin{pmatrix} & * & \\ 0 & \cdots & 0 \\ & * & \end{pmatrix}}_{=0} + (-1)^{l+l}\det I_{n-1} = 1$$

$$\det(E^{\lambda k+l}) \underset{\text{cofactor on } l\text{TH row}}{=} (-1)^{k+k}\det I_{n-1} + (-1)^{k+l}\lambda \underbrace{\det\begin{pmatrix} & * & \\ 0 & \cdots & 0 \\ & * & \end{pmatrix}}_{=0} = 1 \checkmark$$

For general $A \in Mat_{n\times n}$.

If A not invertible $\Rightarrow$ $\det A = 0$

$\Downarrow$

A^t also not invertible (since $rk\, A^t = rk\, A$)

$\Downarrow$

$\det A^t = 0$

If A invertible $\Rightarrow$ $A = E_p \cdots E_1$, each E_i is elementary matrix

$\Downarrow$

$$\det A^t = \det(E_p \cdots E_1)^t$$
$$= \det(\underbrace{E_1^t \cdots E_p^t}_{\text{still elementary matrices}}) = \det E_1^t \cdots \det E_p^t$$
$$= \det E_1 \cdots \det E_p$$
$$= \det E_p \cdots \det E_1$$
$$= \det(E_p \cdots E_1) = \det A$$

QED.

Therefore, cofactor may also takes place at any column:

Thm.
$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \det(\widetilde{A^{ij}})$$

and Thm1. ~ Thm6 are also true when "row" is replaced with "column".

∴ In previous example

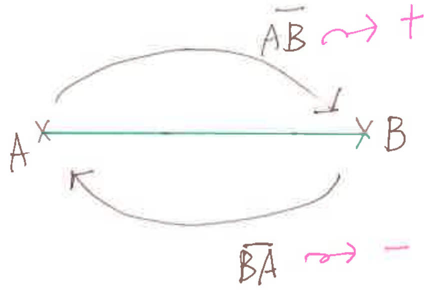
$$\det A = -2 \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \det\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$
$$= -2 \cdot (-2) - (-2) = 6.$$

* Geometric Significance of Determinants

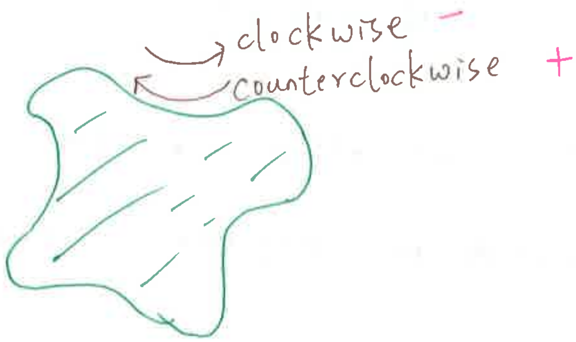
Orientation

An orientation is a consistent way to assign a sign to an bounded n-dimensional object in $\mathbb{R}^n$ characterized by how we travel on the boundary.

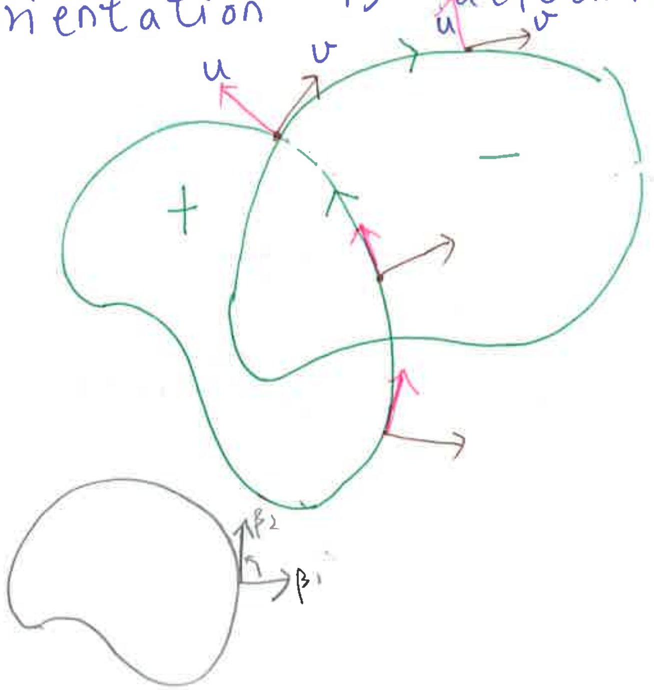
$n=1$:



$n=2$



Orientation is determined by ordering of basis.



$\beta = \{\beta_1, \beta_2\}$ basis for $\mathbb{R}^2$

$\beta_1 = u$, $\beta_2 = v$

$\to$ a "positive" region

$\beta' = \{\beta_1', \beta_2'\}$ basis for $\mathbb{R}^2$

$\beta_1' = v$, $\beta_2' = u$.

$\to$ a "negative" region.

We say that two basos $\beta = \{\beta_1, \beta_2\}$, $\beta' = \{\beta_1', \beta_2'\}$

are in the same orientation if they determine
opposite "

regions of equal
opposite signs

How do we determine?

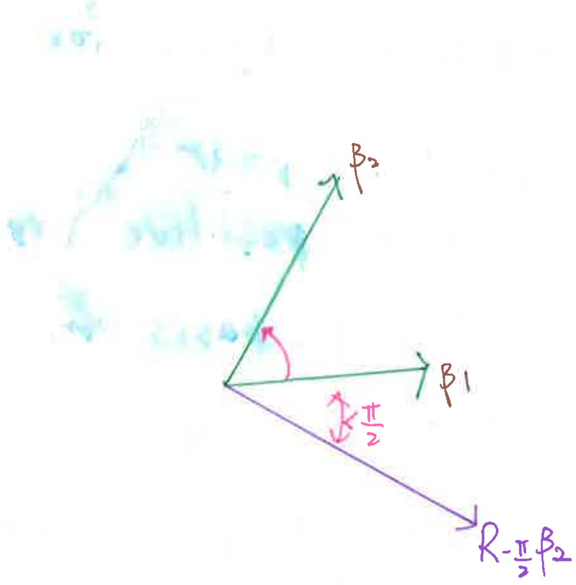
skip.

β and β' are in the same orientation if rotations from β_1 to β_2 and from β_1' to β_2' of angle $\leqslant \pi$ are both counterclock-wise or clockwise

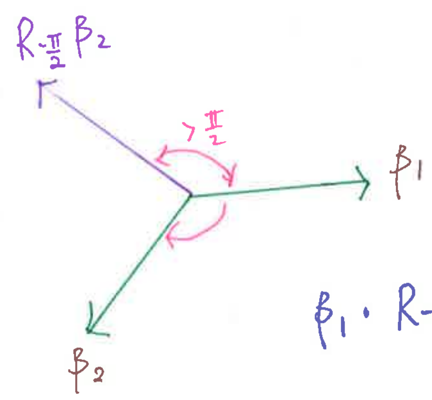
If we set counterclockwise rotation to be positive orientation

$\rightsquigarrow$ clockwise rotation would be negative orientation.

Illustrative Idea



$\beta_1 \cdot R_{-\frac{\pi}{2}}\beta_2 > 0$



$\beta_1 \cdot R_{-\frac{\pi}{2}}\beta_2 < 0$

CONT.,

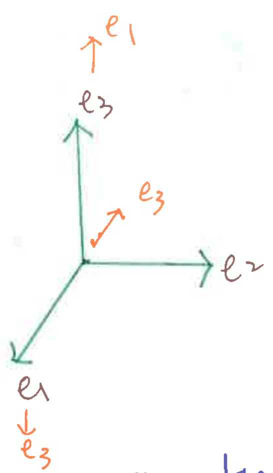
But note that if $\beta_1 = \begin{pmatrix} u \\ v \end{pmatrix}$, $\beta_2 = \begin{pmatrix} w \\ z \end{pmatrix}$

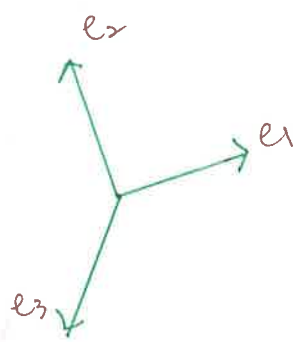
$$\Rightarrow R_{\frac{-\pi}{2}} \beta_2 = \begin{pmatrix} z \\ -w \end{pmatrix} \quad \Rightarrow \quad \beta_1 \circ R_{\frac{-\pi}{2}} \beta_2 = uz - vw = \det(\beta_1 \ \beta_2)$$

$\therefore$ with respect to counterclockwise orientation. (ie. $\{e_1, e_2\}$ taken as "reference orientation") positive (negative) orientation of $\beta = \{\beta_1, \beta_2\}$ is in $\det(\beta_1, \beta_2) > 0$ (< 0).

For $n = 3$, the orientation is determined by "right hand rule":

$\{e_1, e_2, e_3\}$ or any cyclic permutation $\{e_2, e_3, e_1\}$, $\{e_3, e_1, e_2\}$.. of it is referenced to be positive orientation. In all these permutations, the direction of the third vector is determined by the thumb of the right hand when wrapping the fingers from first to second vector.





Switching any two vectors always reverse the orientation.

Therefore, orientation is determined by the sign of the determinant.

We use this universal property to define orientation in n-dimension:

Taking $e=\{e_j\}_{j=1}^n$ as positive reference, a basis $\{\beta_j\}_{j=1}^n$ is in positive orientation $\Leftrightarrow$ $\det(\tilde{\beta}_1, \dots, \tilde{\beta}_n) > 0$.

$\tilde{\beta}_i = (\beta_i)_e$

Given two bases, $\beta = \{\beta_j\}_{j=1}^n$, $\gamma = \{\gamma_i\}_{i=1}^n$, $\exists$ $A \in Mat_{n\times n}$, invertible.

$$(\tilde{\beta}_1 \dots \tilde{\beta}_n) = A(\tilde{\gamma}_1 \dots \tilde{\gamma}_n) \quad \left(\text{ie } A = (\beta_1 \dots \beta_n)(\gamma_1 \dots \gamma_n)^{-1}\right)$$

transition matrix

$\Rightarrow$ $\det(\tilde{\beta}_1 \dots, \tilde{\beta}_n)$ and $\det(\tilde{\gamma}_1 \dots, \tilde{\gamma}_n)$ have $\begin{cases} \text{the same sign} \\ \quad \Leftrightarrow \det A > 0 \\ \text{opposite sign} \\ \quad \Leftrightarrow \det A < 0. \end{cases}$

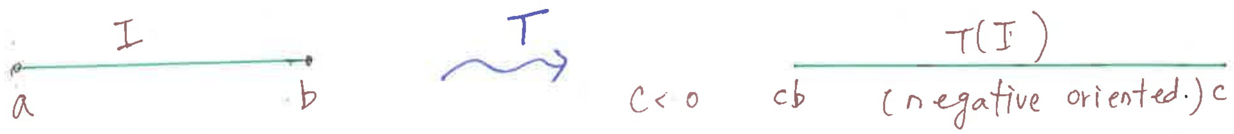
A is "orientation preserving"

A is "orientation reversing"

$T: \mathbb{R} \to \mathbb{R}$ linear, invertible

$T(x) = cx \qquad [T] = (c) \quad , \quad \det[T] = c \neq 0$

Choose + oriented



c<0 cb (negative oriented.) ca

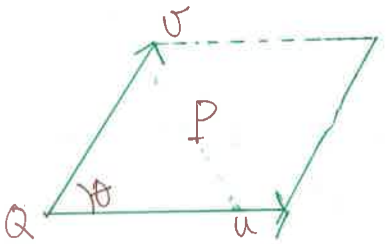
c>0 ca (positive ") cb

$V(I) = b - a > 0$

$$V(T(I)) = c(b-a) \begin{cases} >0 \ ; \text{ if } c>0 & \text{orientation preserved} \\ <0 \ ; \text{ if } c<0 & \text{" reversed.} \end{cases}$$

$= \det[T] \cdot (b-a)$

$n = 2.$



the size of two dimensional object is determined by area of parallelogram by the two spanning vectors

$P = \{Q + tu + sv \mid t, s \in [0,1]\}$

$$\text{Area } P = \|\vec{u}\| \, \|v\| \sin\theta$$
$$= \|u\| \, \|R_{-\frac{\pi}{2}} v\| \cos \qquad = u \cdot R_{-\frac{\pi}{2}} v = \det(u, v)$$

1. $u = \lambda v \rightsquigarrow \text{Area} = 0$

2. $u \to \lambda u$ (OR $v \to \lambda v$) $\rightsquigarrow$ Area $\to \lambda \cdot$ Area

3. $\{u, v\}$ in + orientation $\to \begin{cases} \text{Area} > 0 \\ \text{Area} < 0 \end{cases}$
$\{u, v\}$ " − "

$u \perp v \to \det(u, v) = \pm |u| |v|$

But $\forall\, u, v \in \mathbb{R}^2$,

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad R_{-\frac{\pi}{2}} v = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$$

$$\Rightarrow \quad u \cdot R_{-\frac{\pi}{2}} v = u_1 v_2 - v_1 u_2 = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

$\therefore \quad \beta = \{\beta_1, \beta_2\}$ are in $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$ orientation if $\det \beta \begin{cases} >0 \\ <0 \end{cases}$.

In general, an ordered basis $\beta = \{\beta_1, \dots \beta_n\}$ of $\mathbb{R}^n$ is in positive (negative) orientation if

$$\det(\beta_1 \cdots \beta_n) > 0 \quad (<0)$$

- n-dimensional volume

volume is a measure of how much space an n-dimensional object occupies in $\mathbb{R}^n$.

More commonly known:

$$\left.\begin{array}{lll} n=1 & \rightsquigarrow & \text{length} \\ n=2 & \rightsquigarrow & \text{area} \\ n\geq 3 & \rightsquigarrow & \text{volume} \end{array}\right\} \text{"measure"}$$

Volume has a sign if orientation is taken into account: (signed volume)

$n=1$: Bounded 1-dimensional object in $\mathbb{R}^1$ consists of intervals

$$V(I) = \begin{cases} >0 & \text{if } \underset{b>a}{I} \text{ is } + \text{. oriented} \\ <0 & \text{'' } \underset{b<a}{\text{''}} \text{ '' } - \text{ oriented} \end{cases}$$

+ orient. $a<b$	a	b
− orient. $a>b$	b	a

$$I = \{a + (b-a)t \mid 0 \leq t \leq 1\}$$

Loosely speaking, on $\mathbb{R}^n$, a function

$$V : E^n \to \mathbb{R} \quad , \quad \text{where}$$

$E^n =$ the set of all bases of $\mathbb{R}^n$, satisfying

1~3 is called a volume

[Normalizing, so that $V(\{e_1 \ldots e_n\}) = 1$, we can show that

$$V = \det .]$$

Def'n An (signed) n-dimensional volume of a parellelogram spanned by $u_1, \ldots, u_n$, ie

$$P = \{ s_1 u_1 + \cdots + s_n u_n \mid s_i \in [0,1] \ \forall i \},$$

is $\det(u_1, \ldots, u_n)$.

$\therefore$ consider $T : (\mathbb{R}^n, \beta) \to (\mathbb{R}^n, \beta)$; $\beta = \{e_j\}_{j=1}^n$

$$(u_1', \ldots, u_n') = T(u_1, \ldots, u_n)$$

$\Rightarrow$ volume of parellelogram spanned by $T(u_1), \ldots, T(u_n)$

$= \det [T]_\beta^\beta \cdot$ vol. of parallelogram spanned by $u_1, \ldots, u_n$.