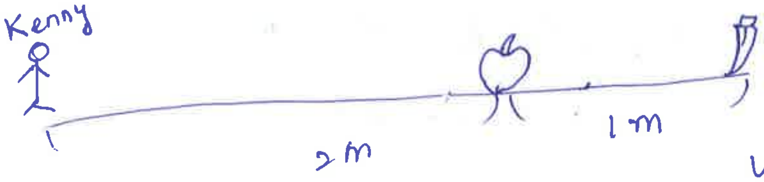
Limit & Continuity

Motivating Question: Q



· walk 1m 1st day and for each day, walk half as much as yesterday.

· Will Kenny ever reach ~~either fruit~~? No!

· Can he be as close to either fruit as he wants? Yes for apple
No for banana. (at least 1m apart)

Most values we compute in calculus are limit of some operations repeated indefinitely.

- explain -

I.

* Limit of a Function.

(Intuitive) Definition of " $\lim_{x\to c} f(x) = L$ "

" as x approaches c, $f(x)$ approaches L "

Loosely defined,

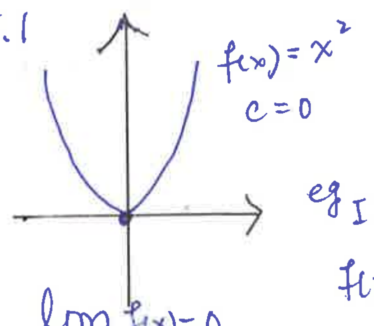
$f(x)$ is as close to L as we want, → depends on how close you want $f(x)$ to L
for all x close enough to c.

(If $c = \infty$ → " " " large enough)

I.0 $f(x) = K$

eg.,

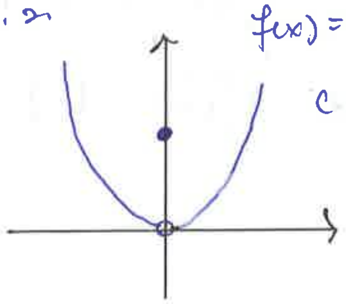
I.1



$\lim_{x\to 0} f(x) = 0$

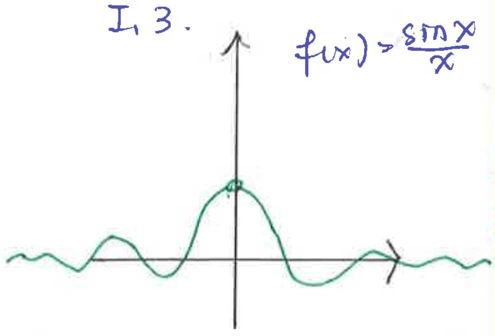
eg I.0 $f(x) = 3$

I.2 $f(x) = \begin{cases} x^2 ; & x \neq 0 \\ 1 ; & x = 0 \end{cases}$ $c = 0$



$\lim_{x\to 0} f(x) = 0$

I.3. $f(x) = \frac{\sin x}{x}$



$\lim_{x\to 0} f(x) = 1$

observe:

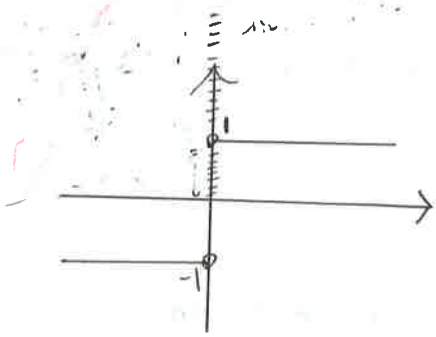
① $f(x)$ doesn't have to be defined at c, and if defined, $f(c)$ is not necessarily $\lim_{x\to c} f(x)$

② computation of limit NEVER assumes $x=c$, but concerns all values of x CLOSE to c.

* $\lim_{x\to c} f(x)$ Doesn't have to Exist!

eg. I.4 $f(x) = \frac{|x|}{x}$ $(\mathbb{R}\setminus\{0\} \to \mathbb{R})$ actually, ± 1

$$= \begin{cases} 1 ; & x>0 \\ -1 ; & x<0 \end{cases}$$



$\lim_{x\to 0} f(x)$ D.N.E.

$\because$ ANY point $x<0$ and

" " $x'>0$ make

$|f(x)-f(x')| = 2$ and so there is no L so that $|f(x)-L|$ is as small as we want for all x close to 0.

Indeed,

$L \geq 0 \Rightarrow |f(x)-L| > L+1$ for ALL $x<0$

$L<0 \Rightarrow |f(x)-L| > |L+1|$ " " $x>0$

However,

$f(x)$ does approach 1 (resp. -1) for all x close enough, too but a little bigger (resp. smaller) than 0. We say,

$f(x)$ approaches 1 (resp. -1), as x approaches 0 from right (resp. left). Denoted. $\lim_{x\to 0^+} f(x) = 1$ (resp. $\lim_{x\to 0^-} f(x) = -1$)

II.

Real Definition of $\lim_{x\to c} f(x) = L$

$f(x)$ defined on $(c-p, c+p)$ except possibly at c. $\&\, p \in \mathbb{R}^+$

$\lim_{x\to c} f(x) = L$ iff, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ s.t. [if $|x-c| < \delta_\varepsilon$, then $|f(x) - L| < \varepsilon$.] $**$

For $\lim_{x\to c^+} f(x) = L$, $**$ only needs to hold for $x > c$

$\lim_{x\to c^-} f(x) = L$. " " " $x < c$

Trivial example

II.1 $f(x) = K$. constant, $c = 4$
$(L = K)$

Little More Interesting

II.2 $f(x) = x$ $(L = 1)$, $c = 1$

II.3 $f(x) = x^2$, $c = 3 \to L = 9$

pick $\delta_\varepsilon = \min(1, \frac{\varepsilon}{4})$

II.4 $f(x) = |x|$, $c = 0$

claim : $L = 0$, since

$$|f(x) - 0| = ||x| - 0| = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$\therefore$ take $\delta_\varepsilon = \varepsilon$,

verify ** : $|x - 0| < \delta_\varepsilon \Rightarrow |x| < \varepsilon \Rightarrow |f(x) - 0| < \varepsilon$

General Strategy : Get relationship b/w ε & δ_ε from relationship b/w $|f(x) - L|$ & $|x - c|$

Observe:

$\lim_{x \to c} f(x) = L$ if and only if

$\lim_{x \to c^-} f(x)$, $\lim_{x \to c^+} f(x)$ exist and $= L$

$\therefore$

In eg I.4,

$$\left.\begin{array}{c} \lim_{x \to 0^-} f(x) = -1 \\ \# \\ \lim_{x \to 0^+} f(x) = 1 \end{array}\right\} \lim_{x \to 0} f(x) \text{ D.N.E.}$$

III. Limit Properties & Computations

Thm 1. Limit is unique, if existed.

$\lim_{x \to c} f(x) = L$ & $\lim_{x \to c} f(x) = M \Rightarrow L = M$

pf: $|L - M| \le |L - f(x)| + |M - f(x)|$ (triangle inequality)

For any $\varepsilon > 0$, there is $\delta_1 > 0$ s.t. $|L - f(x)| < \varepsilon$ for all x in $(c - \delta_1, c + \delta_1)$

$\delta_2 > 0$ " $|M - f(x)| < \varepsilon$ " " $(c - \delta_2, c + \delta_2)$

take $\delta = \min(\delta_1, \delta_2)$ " " $(c - \delta, c + \delta)$,

then for all x in $(c-\delta, c+\delta)$,

$|L - f(x)|$ & $|M - f(x)| < \varepsilon \Rightarrow |L - M| < 2\varepsilon$.

& since ε is arbitrary $\Rightarrow L = M$. #

Thm 2. Limit commutes w/ (almost) all simple algebraic operations:

- $\lim_{x \to c} [\alpha_1 f_1(x) \pm \alpha_2 f_2(x) \pm \cdots \pm \alpha_n f_n(x)]$
  $= \alpha_1 \lim_{x \to c} f_1(x) \pm \cdots \pm \alpha_n \lim_{x \to c} f_n(x)$

  - constants can always be brought out
  - lim. of sum/diff. = sum/diff of limit

- $\lim_{x \to c} [f_1(x) \cdot \cdots \cdot f_n(x)]$
  $= \lim_{x \to c} f_1(x) \cdot \cdots \cdot \lim_{x \to c} f_n(x)$. limit of product = product of limit

- $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$; provided $\lim_{x \to c} g(x) \ne 0$.

pf: exercise. (perhaps do $\lim_{x \to c} (f(x) + g(x))$).

"Nice" Functions: algebraic combinations of familiar elementary functions
$\underbrace{}$ poly's, trig's, exponentials, roots (at $x>0$) ...

without 0 denominator.

Usually, for nice function $f(x)$, $\lim_{x\to c} f(x) = f(c)$
(i.e. Find limit by "plugging in")

eg. $\lim_{x\to 2} e^{x^2+4} = e^{20}$, $\lim_{x\to 9} \cos(x-9) = 1$ ...

$\lim_{x\to 6} \frac{x+4}{x^2-7} = \frac{10}{29}$ ...

*Thm 3. (Limit D.N.E. when $\frac{L\neq 0}{0}$ after "plugging in")

$\lim_{x\to c} f(x) = L \neq 0$ & $\lim_{x\to c} g(x) = 0 \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)}$ D.N.E.

pf. if exist, say $K = \lim_{x\to c} \frac{f(x)}{g(x)}$

$\Rightarrow L = \lim_{x\to c} f(x) = \lim_{x\to c}\left[g(x)\frac{f(x)}{g(x)}\right] = \lim_{x\to c} g(x) \lim_{x\to c} \frac{f(x)}{g(x)}$

$= 0 \cdot K = 0 \quad \rightarrow\leftarrow$

eg. III.1 $\lim_{x\to 0} \frac{1}{x}$ D.N.E. $\lim_{x\to 0^-} f(x) = -\infty$ & $\lim_{x\to 0^+} f(x) = +\infty$. *

Interesting case to compute:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, after "plugging in"

$\lim_{x\to c} \frac{f(x)}{g(x)} \rightarrow \begin{cases} \text{if } f(x) \text{ goes to 0 "faster" than } g(x) \text{ as } x \text{ goes to } c \rightarrow 0 \\ \text{" " " " "slower" " " } \rightarrow \infty \text{ (OR DNE)}, \\ \text{if } f(x) \text{ \& } g(x) \text{ go to 0 proportionally as } x\to c \rightarrow \text{some finite number} \end{cases}$

eg. $(x-c)^3$ vs $(x-c)^2$, e^x vs x^2

eg III 2. $\lim\limits_{x\to 3} \dfrac{x^2-x-6}{x-3} = \lim\limits_{x\to 3} \dfrac{(x+2)\cancel{(x-3)}}{\cancel{x-3}} = 5$ (Factoring)

eg III 3. $\lim\limits_{x\to -3} \left(\dfrac{4x}{x+3} + \dfrac{12}{x+3}\right) = \lim\limits_{x\to -3} \left(\dfrac{4\cancel{(x+3)}}{\cancel{x+3}}\right) = 4$ (Combining)

eg III 4. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1} = \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{(\sqrt{x}-1)(\sqrt{x}+1)}} = 2$

(Rationalization)

⋮

Idea: Play with algebra to get rid of the 0 at bottom.
(OR to get $\frac{L\neq 0}{0}$, in which case we know limit D.N.E.)

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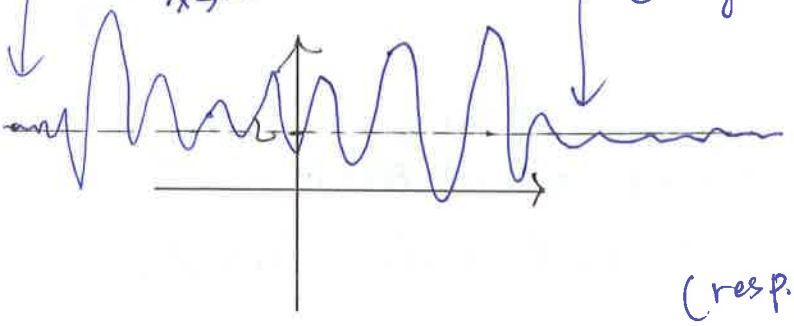
IV Limit at Infinity

"$\lim_{x\to+\infty} f(x) = L$" $\Leftrightarrow$ $f(x)$ is as close to L as we want for all x (positively) large enough "

"$\lim_{x\to-\infty} f(x) = L$" $\Leftrightarrow$ " " " " " (negatively) " "



Precisely, $\lim_{x\to\infty} f(x) = L$ (resp. $\lim_{x\to-\infty} f(x) = L$) if for all $\varepsilon > 0$, there is $M_\varepsilon > 0$ so that for all $x > M_\varepsilon$ (resp. $< -M_\varepsilon$)

$$|f(x) - L| < \varepsilon.$$

eg// IV 1. $f(x) = \frac{1}{x}$; $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

IV 2. $f(x) = \frac{x^3+2}{x^2-1}$; $\lim_{x\to\infty} \frac{x^3+2}{x^2-1} = \lim_{x\to\infty} \frac{x + 2/x^2}{1 - 1/x^2} = \infty$

IV 3. $f(x) = \frac{(x^2-2)^3}{(x^3+2x+1)^2}$; $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^6 + \sum_{i=1}^{5} a_i x^i}{x^6 + \sum_{j=1}^{5} b_j x^j}$

$$= \lim_{x\to\infty} \frac{1 + \sum_{i=1}^{5} a_i x^{i-6} \to 0}{1 + \sum_{j=1}^{5} b_j x^{j-6} \to 0}$$

$i-6$, $j-6$ all < 0

$= 1$

eg. IV 4

$$f(x) = \frac{(\sqrt{x^8} + 3x)^2}{x^6 + 4x + 3} = \frac{x^8 + \text{lower order terms}}{x^6 + \text{lower " terms}} = \frac{x^2 + \sum_{i=1}^{7} a_i x^{-i}}{1 + \sum_{i=1}^{7} b_i x^{-i}}$$

$$\Rightarrow \quad \lim_{x \to \pm\infty} f(x) = \infty$$

When computing limits of rational functions as $x \to \pm\infty$, only consider the most dominating terms on top and bottom.

Order of Dominance (in general)

[Factorial] > exponential e^x, a^x > polynomial x^n, & $x^n > x^m$ if $n > m$ > Oscillating $\cos x$, $\sin x$ ~ constant

eg.

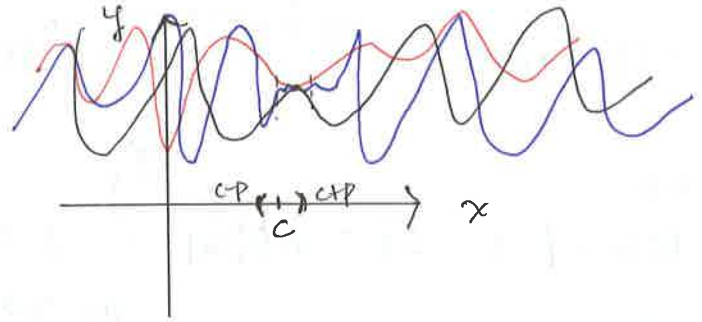
$$f(x) = \frac{x^3 + \cos x}{3x^3} = \frac{1}{3} + \frac{1}{3}\frac{\cos x}{x^3}$$

$$\lim_{x \to \pm\infty} f(x) = \frac{1}{3} + \frac{1}{3} \lim_{x \to \pm\infty} \frac{\cos x}{x^3} = \frac{1}{3}$$

since $-1 \le \cos x \le 1$

V. More Techniques on Limit Computations

* Pinching (Squeeze) Theorem



If for some $p>0$,
$h(x) \le f(x) \le g(x)$
for all x in $(c-p, c+p)$
and $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$
$\Rightarrow \lim_{x\to c} f(x) = L$.

Cor. $\lim_{x\to c} |f(x)| = 0 \Rightarrow \lim_{x\to c} f(x) = 0$

pf. $-|f(x)| \le f(x) \le |f(x)|$

and $\lim_{x\to c} -|f(x)| = -\lim_{x\to c} |f(x)| = 0$ → apply pinching thm.

eg VI: Show $\lim_{x\to 1} \underbrace{(x-1)\sin\frac{1}{x-1}}_{f(x)} = 0$

$\underset{h(x)}{0} \le |f(x)| = |x-1|\left|\sin\frac{1}{x-1}\right| \le \underset{g(x)}{|x-1|}$.

& and apply pinching thm.

Don't Forget:

$-1 \le \sin(*\#\sim☺) \le 1$

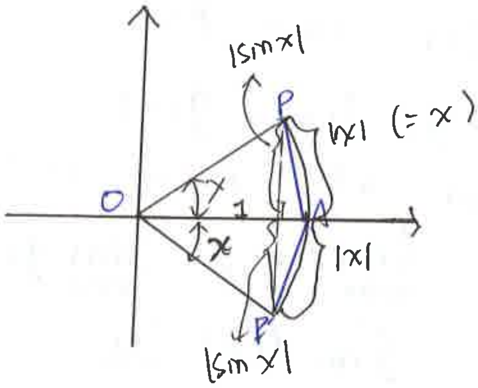
$-1 \le \cos(☆☺\sim☺) \le 1$

$$\boxed{\sin^2(☺) + \cos^2(☺) = 1}$$

$/\sin^2(☺)$: $1 + \csc^2(☺) = \csc^2(☺)$

$/\cos^2(☺)$: $\tan^2(☺) + 1 = \sec^2(☺)$

eg VI 2. Show $\lim_{x\to 0} \sin x = 0$ (forget about "plugging in")



Recall: angle $= \dfrac{\text{arc length}}{\text{radius}}$ (in radian)

For $x \geq 0$

$0 \leq |\sin x| \leq \overline{AP} \leq |x| \;(``\widehat{AP}") \longrightarrow 0$ as $x\to 0$

For $x < 0$

$0 \leq |\sin x| \leq \overline{AP'} \leq |x| \;(``\widehat{AP'}") \longrightarrow 0$

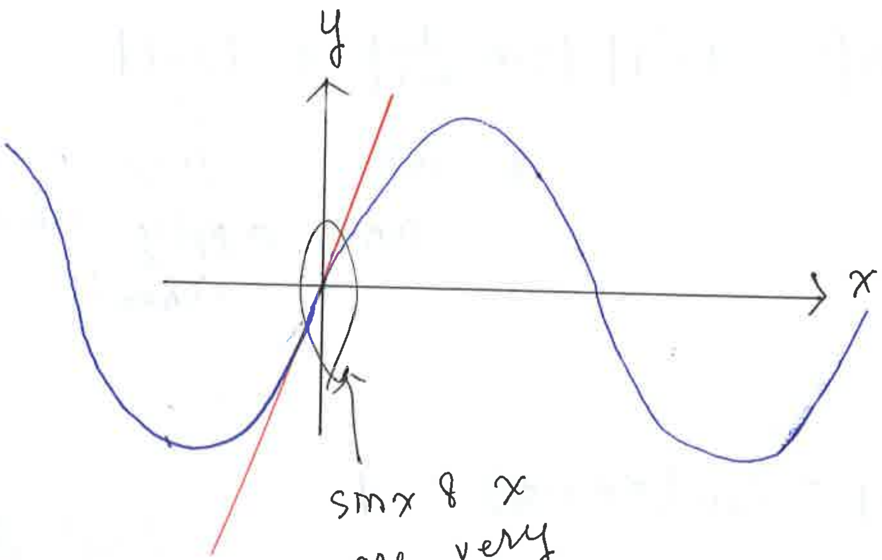
straight line is shortest b/w two points

and apply pinching thm.

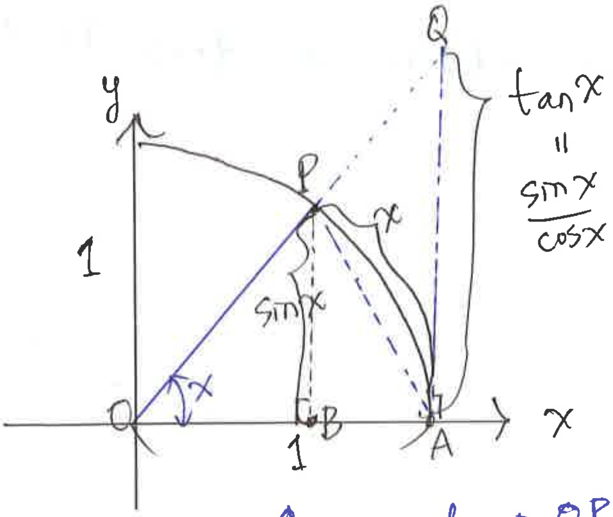
& corollary. #

using $\sin^2 x + \cos^2 x = 1 \Rightarrow \lim_{x\to 0} \cos x = 1$ (not -1, why!?)

Important Identity: $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$



$\sin x$ & x are very close to each other near $x=0$, they approach 0 "equally fast" "pf" Remember: $\sin(-x) = -\sin x$ & $\cos(-x) = \cos x$



Assume $x > 0$

Area of $\triangle OPA$ < Area of sector OPA ≤ Area of $\triangle OQA$

$$\frac{1}{2}\cdot 1^2 \cdot \sin x < \frac{1}{2}\cdot 1^2 \cdot x < \frac{1}{2}\cdot 1^2 \cdot \frac{\sin x}{\cos x}$$

$/(\frac{1}{2}\sin x)$ $\quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \Rightarrow \cos x < \frac{\sin x}{x} < 1$ ... ⊛

⊛ holds for $x<0$ too, since $\cos(-x) = \cos x$

and $\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$

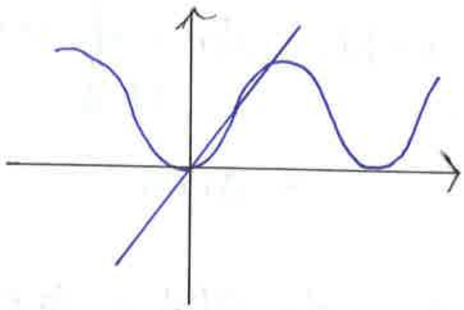
∴ apply pinching thm to ⊛ at (w/ $c=0$)

#

Show :

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

as $x \to 0$, $1-\cos x$ goes to 0 faster than x does.



Pf

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

Try to appeal to $\sin^2 x+\cos^2 x=1$ as much as you can

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x+x\cos x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{\overset{\shortparallel}{1}} \underbrace{\lim_{x\to 0} \frac{\sin x}{1+\cos x}}_{\overset{\shortparallel}{0}} = 0$$

#

Variational Applications

$a \neq 0$

$$\lim_{x\to 0} \frac{\sin ax}{ax} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

$y = ax$

$x \to 0 \Leftrightarrow y \to 0$

$$\lim_{ax} \frac{1-\cos ax}{ax} = \lim_{y\to 0} \frac{1-\cos y}{y} = 0$$

Change of variables

eg V3:

$$\lim_{x\to 0} \frac{\sin 4x}{3x} = \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{4}{3}$$

$$= \frac{4}{3} \lim_{x\to 0} \frac{\sin 4x}{4x} = \frac{4}{3} \;//$$

$$\lim_{x\to 0} \frac{1-\cos 2x}{5x} = \lim_{x\to 0} \frac{1-\cos 2x}{2x} \cdot \frac{2}{5} = 0 \;//$$

eg V4: (playing w/ trig. identity)

$$\lim_{x\to 0} \frac{\sec^2(2x)-1}{x^2} = \lim_{x\to 0} \frac{\tan^2(2x)}{x^2}$$

$$= \lim_{x\to 0} \left[\frac{\sin^2(2x)}{x^2} \cdot \frac{1}{\cos^2(2x)} \right]$$

$$= \left[2 \lim_{x\to 0} \frac{\sin(2x)}{2x} \right]^2 \cdot \lim_{x\to 0} \frac{1}{\cos^2(2x)} = 4 \;//$$

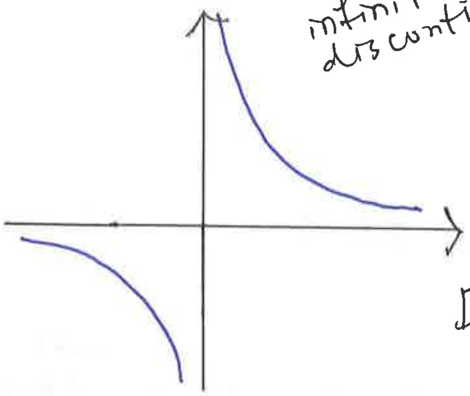
VI. Continuity

Def'n f is "continuous" at c if $\lim_{x\to c} f(x) = f(c)$.

$f: (c-p, c+p) \to \mathbb{R}$

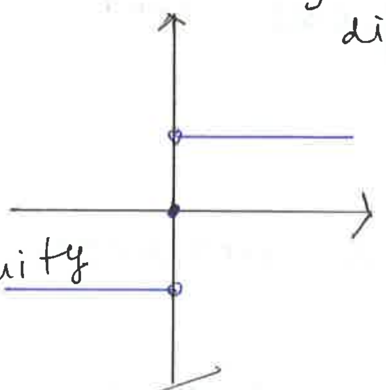
i.e. $\lim_{x\to c} f(x)$ has to exist and has to $= f(c)$, $f(x)$ is as close to $f(c)$ as we wish for all x close enough to c.

infinite discontinuity



Some Types of Discontinuity

Jump discontinuity



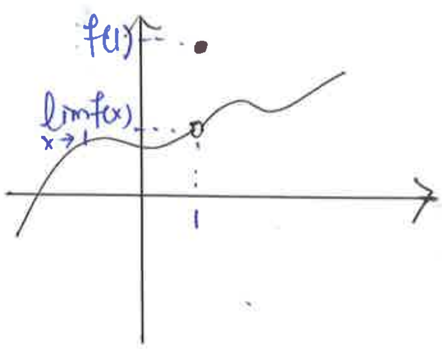
$f(x) = \begin{cases} \frac{x}{|x|} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

not continuous at 0 (limit doesn't exist)

$f(x) = \frac{1}{x}$
Not continuous at $x=0$ (not even defined)

$\underbrace{\hspace{6em}}_{\text{Essential Singularity (not removable)}}$

Removable Singularity (can redefine $f(x)$ to make it cont. at c)



$f(x)$ not continuous at $x=1$

since $\lim_{x\to 1} f(x) \neq f(1)$

Good News: All elementary functions are continuous wherever they are defined.

Thm// If $f(x)$ and $g(x)$ are continuous at $x=c$,

then

$$\alpha f(x) + \beta g(x) \quad \text{is cont. at } c$$

$$f(x)\cdot g(x) \quad \text{" " " "}$$

$$\frac{f(x)}{g(x)} \quad \text{" " " " if } g(c)\neq 0$$

Thm// $f(x)$ cont. at c and $g(x)$ cont. at $g(c)$

$\Rightarrow$ $f\circ g(x)$ cont. at c

eg VI.1 $\sec x$ is continuous at 0 since

$\sec x = f\circ g(x)$ where $g(x) = \cos x$, cont. everywhere

& $f(x) = \frac{1}{x}$, which is cont. at $\cos 0 = 1$