# VII. Spans, Linear Independence, Basis



A vector space = a finite set with $+, \cdot$.

Let $V \subseteq (\mathbb{R}^n, +, \cdot)$ be a subspace. We say

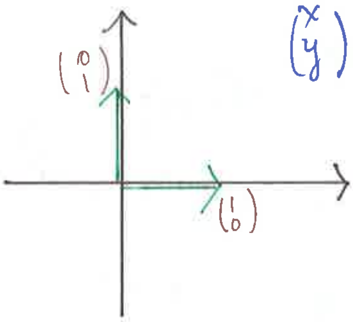
$v_1, \dots, v_m \in V$ spans $W$ if

$Sp(v_1, \dots, v_m) = V$. ($\subseteq$ is automatic, only need $\supseteq$)

i.e. $\forall x \in V$, $x = \alpha_1 v_1 + \dots + \alpha_m v_m$ ; $\alpha_i \in \mathbb{R}$

$n=2$: $V = \mathbb{R}^2 = Sp\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

Ⓐ



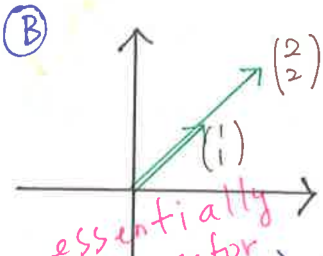
$\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$

$\forall \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$

Ⓑ

$Sp\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} \neq \mathbb{R}^2$

$\begin{pmatrix}1\\0\end{pmatrix} \notin$



essentially one vector only $\Rightarrow$

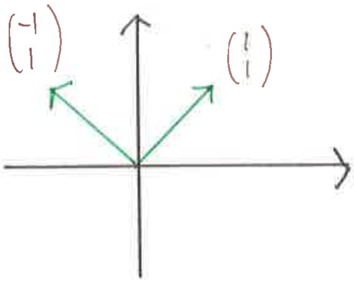
$\to$ too few

$\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}2\\2\end{pmatrix}$

$\begin{cases} a + 2b = 1 \\ a + 2b = 0 \end{cases}$

Ⓓ

$Sp\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\} = \mathbb{R}^2$

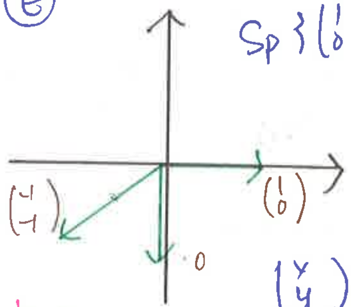


$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}-1\\1\end{pmatrix}$

$\Rightarrow \begin{cases} a = \frac{1}{2}(x+y) \\ b = \frac{1}{2}(y-x) \end{cases}$

Ⓔ

$Sp\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\} = \mathbb{R}^2$



too many vectors

since $\forall \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$

$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\-1\end{pmatrix} + c\begin{pmatrix}-1\\-1\end{pmatrix}$

$\therefore$ Ⓐ

$\begin{cases} a + 0b - c = x \\ 0a - b - c = y \end{cases}$

$\Rightarrow \begin{cases} a = x - y - 2c \\ b = y - c \end{cases}$

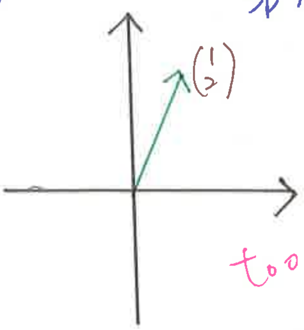
where $c$ is arbitrary

in particular $c = 0$

$\Rightarrow \begin{pmatrix}-1\\-1\end{pmatrix}$ not needed.

Ⓒ

$Sp\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\} \neq \mathbb{R}^2$



since

$\begin{pmatrix}1\\0\end{pmatrix} \notin Sp\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$

too few vectors

observe, in $\textcircled{E}$:

$$\begin{pmatrix}-1\\-1\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} ;$$

ie. $\begin{pmatrix}-1\\-1\end{pmatrix} \in Sp\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}$

& $\textcircled{*} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = (a-1)\begin{pmatrix}1\\0\end{pmatrix} + (b+1)\begin{pmatrix}0\\-1\end{pmatrix}$
$= a'\begin{pmatrix}1\\0\end{pmatrix} + b'\begin{pmatrix}0\\-1\end{pmatrix}$

$a' = x$, $b' = -y$ $\Rightarrow \begin{pmatrix}-1\\-1\end{pmatrix}$ is redundent

$Sp\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\} = Sp\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}$

(in fact, we may remove any of the three vectors without changing the span)

We say that vectors $v_1, \ldots, v_m \in \mathbb{R}^m$ linearly dependent if some $v_i \in Sp\{v_1, \ldots, \hat{v_i}, \ldots, v_m\}$ ($v_i$ skipped)

and linearly independent if they're not dependent. (ie. no $v_i \in Sp(v_1, \ldots, \hat{v_i}, \ldots, v_m)$.)

to introduce at p3

Note: $v_1, \ldots, v_m \neq 0$ linearly independent

$$\Updownarrow$$

$$a_1 v_1 + \cdots + a_m v_m = 0 \Rightarrow a_i = 0 \quad \forall i$$

$\Downarrow$ if $a_1 v_1 + \cdots + a_m v_m = 0$ w/ $a_i \neq 0$

$v_i = \sum_{j \neq i} \frac{a_j}{a_i} v_j \Rightarrow v_i \in Sp\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$
$\therefore$ linearly dependent

$\Uparrow$ if linearly dependent,

$v_i \in Sp\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$

$v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_m v_m$ $\quad \alpha_i$ not all 0

$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} - v_i + \alpha_{i+1} v_{i+1} + \alpha_m v_m = 0$

For $V = Sp(v_1, \dots v_m)$, how many generating vectors
generating vectors
can we take away and still generate $V$ ?

Taking away $v_i$ doesn't change $Sp(v_1, \dots, v_m)$
$\Leftrightarrow Sp(v_1, \dots, v_m) = Sp(v_1, \dots, \hat{v_i}, \dots, v_m)$
$v_i$ skipped.

$\Leftrightarrow v_i \in Sp(v_1, \dots, \hat{v_i}, \dots, v_m)$
$\Leftrightarrow v_1, \dots, v_m$ are linearly dependent

$\therefore$ we may keep taking away generating vectors until the vectors left over are linearly independent, and the set of left over vectors $\{v_1, \dots, v_r\}$ is called a basis of $V$.

Def 11 (Basis)
A basis for a subspace $V$ is a set of linearly independent vectors $v_1, \dots, v_r$ that span $V$.

Examples of basis :

- $\{e_i\}_{i=1}^{n}$ ; where $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← ith , is called the standard basis of $\mathbb{R}^n$. } Basis for unique

- $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

- $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is a basis for hyperplane $H$ with defining equation $x+y+z=0$.

Thm. VI.1 If $\{v_1, .., v_r\}$ is a basis for $V$, then [every vector $v \in V$ can be rewritten uniquely as a linear combination of $v_1, .., v_r$]

→ generalized definition of basis for abstract vector space w/ infinite dimension

pf. $V = Sp(v_1, \dots, v_r)$

$\therefore \forall v \in V$ ; $v = a_1 v_1 + \dots + a_r v_r$.

Suppose $v = a_1' v_1 + \dots + a_r' v_r$.

$\Rightarrow 0 = (a_1 - a_1') v_1 + \dots + (a_r - a_r') v_r$.

Since $v_1, .., v_r$ linearly indep.

$\Rightarrow a_i - a_i' = 0 \ \forall i \quad \Rightarrow a_i = a_i' \ \forall i$

QED

Next we aim to establish the fact that every basis of a vector space $V$ has the same number of vectors. ⑤

Thm. VII 2 Every subset of $Sp(v_1, \dots, v_d)$ with more than $d$ vectors is linearly dependent.

Pf. Let $w_1, \dots, w_r \in Sp(v_1, \dots, v_d)$ with $r > d$.

$\therefore \forall i, \quad w_i = a_{i1} v_1 + \cdots + a_{id} v_d = \sum_{j=1}^{d} a_{ij} v_j$

consider the system of equations

$$(*) \begin{cases} a_{11} x_1 + a_{21} x_2 + \cdots + a_{r1} x_r = 0 \\ a_{12} x_1 + a_{22} x_2 + \cdots + a_{r2} x_r = 0 \\ \vdots \\ a_{1d} x_1 + a_{2d} x_2 + \cdots + a_{rd} x_r = 0 \end{cases},$$

with $r$ equations and $d$ unknowns

$d > r$, $\Rightarrow$ some unknown is free.

$\therefore \exists$ solution $(c_1, \dots, c_r) \neq (0, \dots, 0)$ solving $(*)$

$$\Rightarrow c_1 w_1 + c_2 w_2 + \cdots + c_r w_r$$
$$= (a_{11} c_1 + a_{21} c_2 + \cdots + a_{r1} c_r) v_1$$
$$+ (a_{12} c_1 + a_{22} c_2 + \cdots + a_{r2} c_r) v_2$$
$$+ \vdots$$
$$+ (a_{1d} c_1 + a_{2d} c_2 + \cdots + a_{rd} c_r) v_d = 0$$

$\therefore w_1, \dots, w_r$ linearly dependent QED.

It immediately implies that every subset of linearly independent vectors in $Sp(v_1, \dots, v_d)$ must have no more than $d$ elements

Thm. VII.3 All bases of a (finite dimensional) subspace have the same number of elements.

Pf. Suppose $\{v_1, \dots, v_d\}$ & $\{w_1, \dots, w_r\}$ are basis for subspace $W$. $(W = Sp(v_1, \dots, v_d) = Sp(w_1, \dots, w_r))$

$\{v_1, \dots, v_d\}$ linearly indep. & $\subseteq Sp\{w_1, \dots, w_r\}$

$$\Rightarrow d \le r$$

$\{w_1, \dots, w_r\}$ " " & $\subseteq Sp\{v_1, \dots, v_d)$

$$\Rightarrow r \le d$$

$\Rightarrow \quad d = r$ QED.

The number $d$ (or $r$) is called the dimension of $W$, or $\dim W$.

A vector space w/ $\dim < \infty$ can be viewed as a finite set w/ ($\dim W$) elements with $+$, $\cdot$

e.g. $\dim \mathbb{R}^n = n$, since $\{e_i\}_{i=1}^n$ is a basis

$\dim H = n-1$, for a hyperplane in $\mathbb{R}^n$

$\dim(\mathbb{R}q) = 1$, since it has basis $\{q\}$

Cor. VII.4 If $m > n$, every set of $m$ vectors in $\mathbb{R}^n$ are linearly dependent.

Convention : 1. Basis can't contain 0. vector, since $0 \in$ Span of any vector(s). and can always be removed.

2. The basis for $\{0\}$ is $\phi$.

* Existence of Basis.

Thm. VIII 5 Every subspace of $\mathbb{R}^n$ has a basis.

pf. Let $W \subseteq \mathbb{R}^n$ be subspace

$W = \{0\} \to$ Basis $= \phi$ ✓

$W \neq \{0\}$.

consider $\mathbb{B} = \{ B \subseteq W \mid$ elements of $B$ are lin. independent $\}$.

$\mathbb{B} \neq \phi$, since $\{w\} \in \mathbb{B}$ $\forall$ $w \in W \setminus \{0\}$.

and clearly every $B$ in $\mathbb{B}$ has no more than $n$ elements (Thm. VII 2.)

Let $\{w_1, \dots w_d\} \in \mathbb{B}$ be a set w/ maximal number of elements.

If $Sp(w_1, \dots, w_d) \neq W$, $\exists$ $v \in W$ s.t. $v \notin Sp(w_1, \dots, w_d)$

$\Rightarrow \{w_1, \dots, w_d, v\}$ linearly independent

$\Rightarrow \{w_1, \dots, w_d, v\} \in \mathbb{B}$ with $d+1$ elements $\rightarrow\leftarrow$

$\therefore$ $\{w_1, \dots, w_d\}$ is a basis for $W$ QED.

By Thm VII 3, $\dim W = d$.,

A basis for $W$ is a set with most possible linearly independent vectors.

Construction of basis for $W \neq \{0\}$

From below:

take $0 \neq w \in W$ and keep adding $w_1, w_2, \dots$ with the restriction that resulting set is linearly indep.

Keep adding until $\{w, w_1, \dots, w_{d-1}\}$ spans $W$

From above:

Given $W = Sp\{w_1, \dots w_r\}$, keep deleting generating vectors until resulting set of generating vectors are linearly independent.

Next, observe that dimension determines whether a subspace is proper (of it's entire space)

Thm VII 6

$V \subseteq W$, both subspaces of $\mathbb{R}^n$. Then

$\dim V \leq \dim W$, and "=" $\Leftrightarrow$ $V = W$.

Pf:: let $\{v_1, \dots, v_c\}$ be basis of $V$ ($\dim V = c$)

$\{w_1, \dots, w_d\}$ " " " $W$ ($\dim W = d$)

$v_1, \dots, v_c$ linearly independent in $V$, and so in $W$. Thm VII. $\Rightarrow$ $c \leq d$.

For second assertion, we prove equivalently $c < d \Leftrightarrow V \neq W$

$V \neq W \Rightarrow \exists\, w \in W \setminus Sp\{v_1, \dots, v_c\}$

$\Rightarrow \{v_1, \dots, v_c, w\}$ linearly indep. in $W$

$\Rightarrow c+1 \leq d$ OR $c < d$.

$c < d \Rightarrow \{v_1, \dots, v_c\}$ not basis for $W$ (all basis has same # of elements)

$\Rightarrow V \neq W$.

QED.

eg.

$$V = Sp\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$\leftarrow$ dim 3

$$a\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\begin{cases} 2a + 0b + c = 0 & \cdots ① \\ a + 0b + c = 0 & \cdots ② \\ a + b + c = 0 & \cdots ③ \end{cases}$$

③ − ② $\Rightarrow$ $b = 0$ $\quad \begin{cases} 2a + c = 0 \\ a + c = 0 \end{cases} \Rightarrow a = 0$

$\Rightarrow c = 0$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ linearly indep.

dim(LHS) = 3 = dim $\mathbb{R}^3$.

# * Coordinate Representation

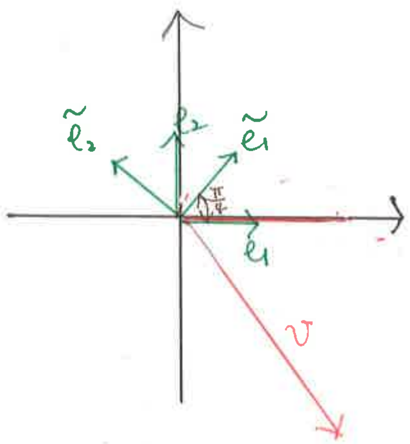
Given $\beta = \{v_i\}_{i=1}^n$, a basis of an $n$-dimensional vector space $V$

$\forall v \in V$, $\exists!$ $a_1, \dots, a_n \in \mathbb{R}$ s.t. (unique)

$$v = a_1 v_1 + \dots + a_n v_n$$

$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is the coordinate representation of $v$ with respect to $\beta$, usually written $(v)_\beta$.

Various $\beta$ gives various coordinate representation




$V = \mathbb{R}^2$ $\qquad$ $\tilde{e}_i = R_{\frac{\pi}{4}} e_i$

$\beta = \{e_1, e_2\}$, $\quad \beta' = \{\tilde{e}_1, \tilde{e}_2\}$

$v = 2e_1 - 3e_2$, $\quad (v)_\beta = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

What is $(v)_{\beta'}$ ?

i.e. $v = ?\,\tilde{e}_1 + ?\,\tilde{e}_2$

We know $v = 2e_1 - 3e_2$

$e_1 = a\tilde{e}_1 + b\tilde{e}_2$ ; $R_{-\frac{\pi}{4}} e_1 = a e_1 + b e_2$

ie $\left(R_{-\frac{\pi}{4}} e_1\right)_\beta = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$

$\therefore (e_1)_{\beta'} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$

Similarly $\quad e_2 = c\tilde{e}_1 + d\tilde{e}_2$

$(R_{\frac{-\pi}{4}} e_2) \qquad R_{\frac{-\pi}{4}} e_2 = c e_1 + d e_2$

$$(R_{-\frac{\pi}{4}} e_2) = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$e_2 = -\frac{\sqrt{2}}{2}\tilde{e}_1 + \frac{\sqrt{2}}{2}\tilde{e}_2, \text{ ie } (e_2)_{\beta_1} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\therefore \quad v = 2e_1 \quad e_2$$

$$= 2\left(\frac{\sqrt{2}}{2}\tilde{e}_1 - \frac{\sqrt{2}}{2}\tilde{e}_2\right) - 3\left(\frac{\sqrt{2}}{2}\tilde{e}_1 + \frac{\sqrt{2}}{2}\tilde{e}_2\right)$$

$$= \frac{-1}{2}\sqrt{2}\,\tilde{e}_1 - \frac{5}{2}\sqrt{2}\,\tilde{e}_2$$

$$\text{ie.} \quad (v)_{\beta'} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{-5}{2}\sqrt{2} \end{pmatrix}$$

check!

In general, given two bases

$$\beta = \{\beta_i\}_{i=1}^{n} \quad \text{and} \quad \beta' = \{\beta'_i\}_{i=1}^{n}$$

For each $i$, solve $\beta_i = \alpha^i_1 \beta'_1 + \cdots + \alpha^i_n \beta'_n = \sum_{j=1}^{n} \alpha^i_j \beta'_j$

$$(v)_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \iff v = a_1\beta_1 + \cdots + a_n\beta_n$$

$$= \sum a_i \beta_i$$

$$= \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n} \alpha^i_j \beta'_j \right)$$

$$= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_i \alpha^i_j \right) \beta'_j$$

$$\iff (v)_{\beta'} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where $b_j = \sum_{i=1}^{n} a_i \alpha^i_j$

In terms of matrix.

$$(v)_{\beta'} = (\gamma_{ij})(a_i)_\beta \quad , \quad \text{where } \gamma_{ij} = \alpha^i_j$$

determined by

Change of Coordinate, 11/13 14 Ends ... in is

determined by change of coordinate on basis elements.

Orthogonal vectors are linearly independent.

$\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ with $v_i \cdot v_j = 0$ if $i \neq j$

and $v_i \neq 0 \ \forall i$

$$a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0$$

$\cdot\, v_1$ : $\quad a_1 \underbrace{v_1 \cdot v_1}_{} {}^{>0} + a_2 \underbrace{v_2 \cdot v_1}_{0} + \underbrace{\dots}_{0} + a_m \underbrace{v_m \cdot v_1}_{0} = 0$

$$\Rightarrow \quad a_1 = 0$$

$\cdot\, v_2 \Rightarrow a_2 = 0 \quad \dots \quad , \quad \cdot\, v_m \Rightarrow a_m = 0.$