IX. Systems of Linear Equations

$m \times n$ system of linear equations is a collection of m linear equations w/ n-unknowns

$$\circledast \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

In matrix form

coefficient matrix $\to$ $A = (a_{ij}) \in Mat_{m\times n}$; $1 \le i \le m$, $1 \le j \le n$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

$\circledast$ may be re-written as

$$Ax = b.$$

the system is homogeneous (inhomogeneous) if $b = 0$ $(b \neq 0)$

General Discussions:

1. $K_b := \{ u \in \mathbb{R}^n \mid Au = b \}$
   $=$ solution for $Ax = b$

K_b is a subspace $\iff$ $b = 0$

$\Rightarrow$ K_b a subspace, $0 \in K_b \Rightarrow A \cdot 0 = b$ ✓
$ \overset{\shortparallel}{0}$

$\Leftarrow$ $K_0 = \{ u \in \mathbb{R} \mid Au = 0 \} = N(L_A)$, ✓ a subspace of $\mathbb{R}^n$ and $\dim K_0 = n - \mathrm{rank}(A)$

2. $0 \in K_0$, $\therefore \{0\} \subset K_0$.

But if $\{0\} = K_0 \Rightarrow L_A$ is injective

If, in addition, that $n = m$, L_A is isomorphism.

L_A is isomorphism $\Leftrightarrow$ $A \in Mat_{n\times n}$ and the only solution to $Ax = 0$ is $x = 0$.

3.

Number of solutions ?

Thm// $Ax = b$ either has no solution, 1 solution, or infinitely many soutions.

Pf// if $x \neq y$ both satisfy $Ax = Ay = b$,

then, $\forall \lambda \in \mathbb{R}$,

$$\begin{aligned} A(\lambda x + (1-\lambda)y) &= A(\lambda x) + A((1-\lambda)y) \\ &= \lambda A(x) + (1-\lambda)Ay \\ &= \lambda b + (1-\lambda)b = b. \end{aligned}$$

$\therefore$ ∞ - many solutions. QED//

Thm// $A \in Mat_{m\times n}$, $K_b = \{x \in \mathbb{R}^n \mid Ax = b \in \mathbb{R}^m\}$

$K_0 = \{x \in \mathbb{R}^n \mid Ax = 0 \in \mathbb{R}^m\}$.

If $K_b \neq \phi$, $K_b = \{s\} + K_0$, where $s \in K_b$ arbitrary.

(a particular solution)

pf// $\subseteq$: $w \in K_b$. $Aw = b$

$$\Rightarrow \underset{\underset{\textstyle A(w-s)}{\parallel}}{Aw - As} = 0$$

$\Rightarrow w - s \in K_0 \Rightarrow w \in \{s\} + K_0$ ✓

$\supseteq$ $s + \underset{\underset{\textstyle K_0}{\in}}{z} \in \{s\} + K_0$

$A(s+z) = As + \overset{=0}{Az} = b$

$\therefore s + z \in K_b \quad \forall z \in K_0$ ✓ QED//

∴ If $Ax = b$ has solution.

"$\dim K_b$" $= \dim K_0$

(i.e. if $\{z_1, \ldots, z_\ell\}$ basis for K_0,

any element of K_b is of the form $s + u$, $u \in Sp(z_1, \ldots, z_\ell)$)

Finding K_b: (solving $Ax=b$; $A\in Mat_{m\times n}$, $b\in\mathbb{R}^m$)

- $m=n$ and A is invertible. $\Rightarrow$ $x = A^{-1}b$

  $K_b=\{A^{-1}b\} \underset{\text{if } b=0}{=} \{0\}$

- if $B\in Mat_{m\times m}$ invertible $K_b = K'_{Bb}$.

  where $K'_{Bb}=\{u\in\mathbb{R}^n \mid (BA)u=Bb\}$

  since $BAx=Bb$

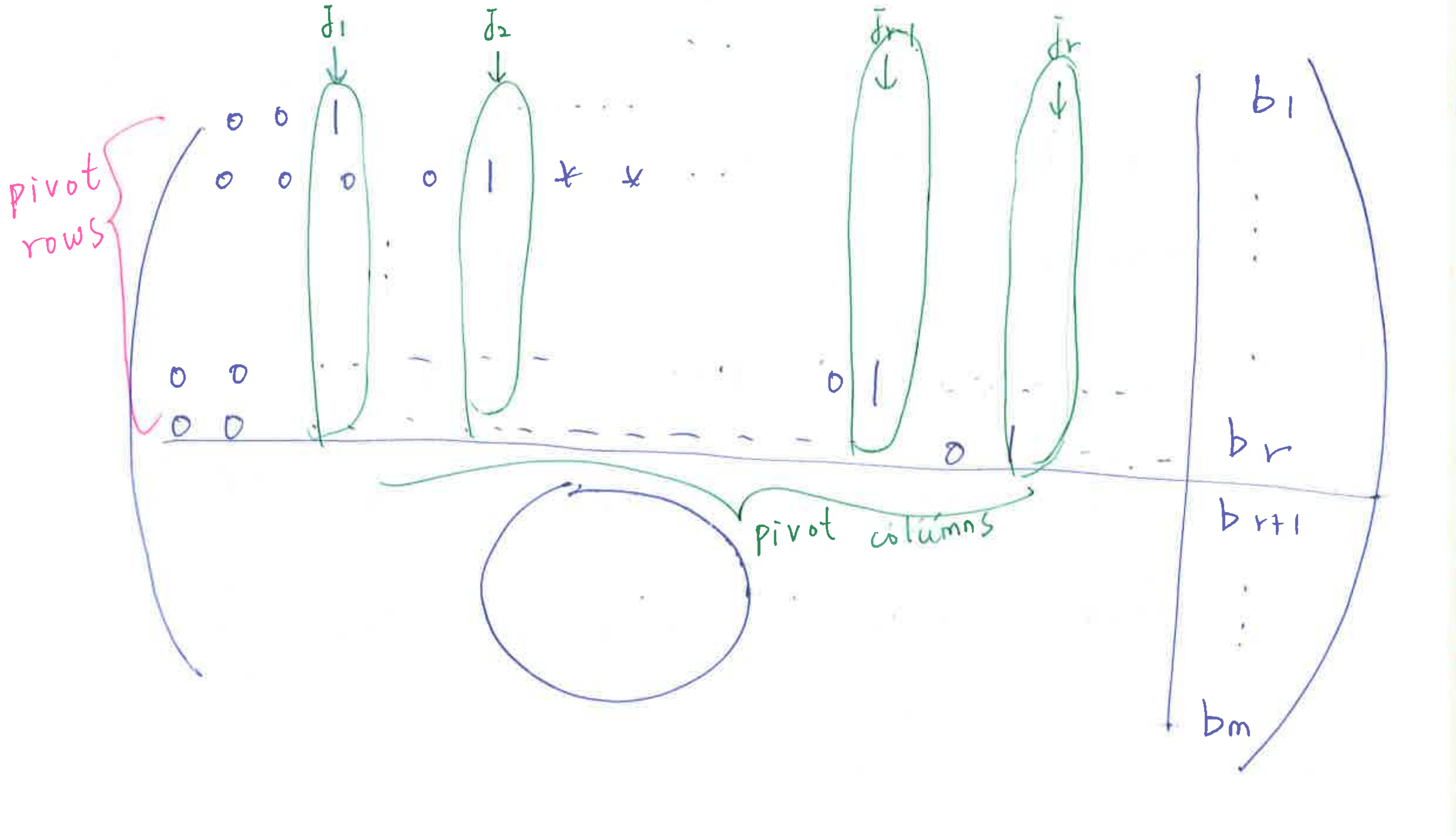
  $x\in K_b \Rightarrow Ax=b \Rightarrow BAx=Bb. \Rightarrow x\in K'_{Bb}$

  $x\in K'_{Bb} \Rightarrow BAx=Bb \Rightarrow B^{-1}(BAx)=B^{-1}(Bb)$

  $\Rightarrow Ax=b \Rightarrow x\in K_b.$

let $(A|b)\in Mat_{m\times(n+1)}$ be the augmented matrix

K_b can be easily described if A is in rref:



if any $b_{r+1}, \ldots, b_m \neq 0 \Rightarrow$ no solution

$(\because\ 0x_1 + 0x_2 + \cdots + 0x_n \neq 0$ not possible $)$

system is called inconsistent.

$K_b = \phi$

eg.

$$\begin{cases} 2x + 4y = 2 & \cdots ① \\ 4x + 8y = 7 & \cdots ② \end{cases} \longleftrightarrow \left(\begin{array}{cc|c} 2 & 4 & 2 \\ 4 & 8 & 7 \end{array}\right)$$

$-2 \cdot ① + ②$:

$$\begin{cases} 2x + 4y = 2 \\ 0x + 0y = 3 \end{cases} \longleftrightarrow \left(\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 0 & 3 \end{array}\right)$$ inconsistent.

$\therefore$ Suppose $Ax = b$ is consistent, ie $b_{r+1} = \cdots = b_m = 0$.

Get r equations:

$$\begin{cases} x_{j_1} + a_{1j_1+1} x_{j_1+1} + \cdots + a_{1n} x_n = b_1 \\ x_{j_2} + a_{1j_2+1} x_{j_2+1} + \cdots + a_{2n} x_n = b_2 \\ \vdots \\ x_{j_r} + a_{rj_r+1} + \cdots + a_{rn} x_n = b_r \end{cases}$$

eq r: $x_{j_r+1}, \ldots, x_n$ free and $x_{j_r} = b_r - \sum_{i=1}^{n-j_r} a_{r, j_r+i}\, x_{j_r+i}$

eq $r-1$: $x_{j_{r-1}+1}, \ldots, \widehat{x_{j_r}}, \ldots, x_n$ free " $x_{j_{r-1}} = b_{r-1} - \sum_{i=1}^{n-j_{r-1}} a_{r, j_{r-1}+i}\, x_{j_{r-1}+i}$

$\vdots$

$$K_b = = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{j_1} \\ x_{j_1+1} \\ \vdots \\ x_{j_2} \\ x_{j_2+1} \\ \vdots \\ x_{j_r} \\ x_{j_r+1} \\ \vdots \\ x_r \end{pmatrix} \;\middle|\; \begin{array}{l} \text{all } x_j \text{ free except } x_{j_i} \\ x_{j_i} \in Sp\{x_{j_i+1}, \dots, x_r\} \end{array} \right\}$$

If $b = 0$, K_0 is a subspace of $\mathbb{R}^n$ with dimension $n - r$. (since all x_j except $x_{j_1}, \dots, x_{j_r}$ are free)

eg11

$$\begin{cases} 0x_1 + x_2 + 2x_3 + 2x_4 + x_5 = 1 \\ 0x_1 + 0x_2 + 0x_3 + x_4 + 2x_5 = 2 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 = 3 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccccc} & 2 & & 4 & 5 & \\ & \downarrow & & \downarrow & \downarrow & \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$x_5 = 3$

$x_4 + 2x_5 = 2 \Rightarrow x_4 = -4$

$x_2 + 2x_3 + 2x_4 + x_5 = 1 \Rightarrow x_2 = 1 - 2x_3 + 8 - 3 = 6 - 2x_3$

x_1, x_3 free

$$K_b = \left\{ \begin{pmatrix} x_1 \\ 6 - 2x_3 \\ x_3 \\ -4 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 3 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \underbrace{\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 3 \end{pmatrix} \right\}}_{\text{“}\{s\}\text{”}} + \underbrace{\mathrm{Sp}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}}_{K_0}$$

For a general system $Ax=b$, we row reduce A into $\mathrm{rref}(A)$, i.e. apply series of row operations

$$E = E_1 \cdots E_p \quad \text{s.t.} \quad EA = \mathrm{rref}(A)$$

In terms of augmented matrix.

$$E(A|b) = (EA|Eb) = (\mathrm{rref}(A)|Eb)$$

Since E is invertible, $K_b = K'_{Eb}$ and we describe K_b by describing $K_{EA=Eb}$.

In particular, if $b=0$ (homogeneous system),

$$(A|0) = (EA|0)$$

null space of A = null space of EA.

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$$\begin{cases} 3x_1 + 2x_2 + 3x_3 - 2x_4 = 1 \\ x_1 + x_2 + x_3 = 3 \\ x_1 + 2x_2 + x_3 - x_4 = 2 \end{cases}$$

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 & 2 \end{array}\right) \xrightarrow{E^{13}\cdot} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 2 & 3 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{E^{-1\cdot1+2}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 3 & 2 & 3 & -2 & 1 \end{array}\right) \xrightarrow{E^{-3\cdot1+3}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right)$$

$$\xrightarrow{E^{-4\cdot2+3}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 & -9 \end{array}\right) \xrightarrow{E^{-\frac{1}{3}\cdot3}\cdot E^{-1\cdot2}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & x_3 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \end{array}$$

x_3 free

$x_4 = 3$

$x_2 - x_4 = -1 \Rightarrow x_2 = 2$

$x_1 + 2x_2 + x_3 - x_4 = 2$

$\Rightarrow \quad x_1 = -2x_2 - x_3 + x_4$

$= -4 - x_3 + 2 = -2 - x_3$

$$K_b = \left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$