IX. Techniques of Integration

In this chapter, we study integrals of more special functions. We, however, do not have formulae to integrate most functions.

* Integration By Parts

Recall product rule:

$$(uv)' = uv' + vu'$$

$\int$ both sides:

$$uv = \int uv'\,dx + \int u'v\,dx$$

or.

$$\int \underbrace{uv'\,dx}_{dv} = uv - \int \underbrace{u'v\,dx}_{du}$$

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du$$

(No need to include constant C here, they will appear after performing $\int v\,du$)

For definite integrals,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

eg., Evaluate $\int xe^x\,dx$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$\therefore \int x e^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C \quad //$$

Choices of u and dv are important:

In the previous example, if

$$u = e^x \quad , \quad dv = x\,dx$$

$$du = e^x\,dx \quad , \quad v = x^2/2.$$

$$\Rightarrow \quad \int x e^x\,dx = \frac{x^2}{2}e^x - \underbrace{\int \frac{x^2}{2}e^x\,dx}_{\text{worse than original integral.}}$$

In general, u = something that becomes nicer after differentiation

dv = " " doesn't become too much worse after integration.

eg. Evaluate $\int x^2 e^{-x}\,dx$

$$u = x^2 \quad , \quad dv = e^{-x}\,dx$$

$$du = 2x\,dx \quad , \quad v = -e^{-x}.$$

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\int x e^{-x}\,dx \quad \cdots \quad ①$$

$$u = x \qquad dv = e^{-x}\,dx$$

$$du = dx \qquad v = -e^{-x}$$

$$① = -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right]$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

//

eg,, Evaluate

$$\int x \cos x \, dx$$

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C \;//$$

eg,, Evaluate

$$\int e^x \cos x \, dx$$

$u = e^x \qquad dv = \cos x \, dx$

$du = e^x \qquad v = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \quad \cdots \text{①}$$

$u = e^x \qquad dv = \sin x \, dx$

$du = e^x dx \qquad v = -\cos x$; plug in RHS ①

$$\text{①} = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\Rightarrow 2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\text{OR} \int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C \;//$$

eg,, Evaluate $\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= x \ln x - x + C \;//$$

* Powers and Products of Trigonometric Functions

Recall basic trig. identities about sum/product:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\cos\beta$$

$\alpha = \beta \rightsquigarrow$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$\sin^2\alpha + \cos^2\alpha = 1$

$$= 2\cos^2\alpha - 1 \quad (*)$$

$$= 1 - 2\sin^2\alpha$$

(*) also gives

$$\left.\begin{aligned}\cos^2\alpha &= \tfrac{1}{2}\left[1 + \cos(2\alpha)\right] \\ \sin^2\alpha &= \tfrac{1}{2}\left[1 - \cos(2\alpha)\right]\end{aligned}\right\}$$

half-angle identities
power reducing tool

- sine and cosine.

Try to evaluate $\int \sin^n x \cos^m x \, dx$

Basic case $n=2, m=0$ (or $m=2, n=0$)

$\rightsquigarrow$ apply half angle identities directly.

$$\int \sin^2 x \, dx = \frac{1}{2}\int\left[1 - \cos(2x)\right]dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \cos(2x)\,dx \overset{u=2x}{=} \frac{x}{2} - \frac{1}{4}\sin(2x) + C \;//$$

and similarly, $\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$ //

General cases:

case 1: one of n, m odd: (say n)

$n-1$ even $\rightsquigarrow$

$$\sin^n x \cos^m x = \sin^{n-1}x \cos^m x \sin x$$

$$= (\sin^2 x)^{\frac{n-1}{2}} \cos^m x \sin x$$

$$= \underbrace{(1 - \cos^2 x)^{\frac{n-1}{2}} \cos^m x}_{\text{a poly of } \cos x} \sin x$$

$\rightsquigarrow$ let $u = \cos x$

eg.

(b)

$$\int \sin^5 x \cos^4 x \, dx$$

$$= \int \sin^4 x \cos^4 x \sin x \, dx$$

$$= \int (\sin^2 x)^2 \cos^4 x \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

let $u = \cos x$

$$= \int (1-u^2)^2 u^4 \, du = \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{u^5}{5} + \frac{2}{7}u^7 + \frac{u^9}{9} + C = \frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x + \frac{1}{9}\cos^9 x + C.$$

(similar methods apply for odd m)

case 2: n, m both even:

need to discuss the case $n=0$ (or $m=0$), since for n, m both even, we may rewrite

$$\sin^n x \cos^m x = (\sin^2 x)^{\frac{n}{2}} \cos^m x$$
$$= (1-\cos^2 x)^{\frac{n}{2}} \cos^m x$$
$$= \text{sum of of } \cos^j x\text{'s with } j \text{ even}$$

For $\cos^m x$, with m even,

$$\cos^m x = (\cos^2 x)^{\frac{m}{2}} = \underbrace{\left(\tfrac{1}{2}[1+\cos(2x)]\right)^{\frac{m}{2}}}$$

Sum of terms $\cos^j ax$'s w/ j even & $\le \frac{m}{2}$

→ may apply half-angle identity to further reduce the power until we can integrate.

eg// $\int \cos^4 x \, dx$

$$= \int (\cos^2 x)^2 dx = \int \left(\tfrac{1}{2}(1+\cos(2x))\right)^2 dx$$
$$= \tfrac{1}{4}\int [1 + 2\cos(2x) + \cos^2(2x)] \, dx$$
$$= \tfrac{1}{4}[x + \sin(2x)] + \tfrac{1}{4}\int \cos^2(2x) \, dx$$
$$= \tfrac{1}{4}[x + \sin(2x)] + \tfrac{1}{4}\int \tfrac{1}{2}[1 + \cos(4x)] \, dx$$
$$= \tfrac{1}{4}[x + \sin(2x)] + \tfrac{1}{8}[x + \tfrac{1}{4}\sin(4x)] + C \quad //$$

Similar techniques apply to $\sin^n x$.

Tangent (cotangent) and Secant (Cosecant)

$$\int \tan^n x \sec^m x \, dx$$

Key formula : $\tan^2 x = \sec^2 x - 1$

Most likely u substitution :

$u = \tan x \rightarrow$ leave $du = \sec^2 x \, dx$ at the end.

$u = \sec x \rightarrow$ leave $du = \sec x \tan x \, dx$ at the end.

pick one and see if it works ...

eg// $\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \sec x \tan x \, dx$

$$u = \sec x \; \stackrel{\downarrow}{=} \int (\tan^2 x)^2 \underbrace{\sec^2 x}_{u^2} \underbrace{\sec x \tan x \, dx}_{du}$$
$$= \int (\sec^2 x - 1)^2 u^2 \, du = \int (u^2 - 1)^2 u^2 \, du$$
$$= \int (u^6 - 2u^4 + u^2) \, du = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C \quad //$$

There are, of course, occasional special cases ...

$$\int \sec^3 x \, dx \stackrel{\uparrow}{=} \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$u = \sec x$; $dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$; $v = \tan x$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\rightsquigarrow \quad 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$= \sec x \tan x + \ln |\sec x + \tan x| + C$$

$$\rightarrow \quad \int \sec^3 x \, dx = \frac{1}{2} \left[\qquad \qquad \qquad \right]$$

csc & cot follow similar techniques.

* Trig. Substitutions.

We integrate functions involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ by playing with square sum/difference identities in trig.

For ...	Let ...	With ...	Turns the $\sqrt{\cdot}$ into ...
$\sqrt{a^2-x^2}$	$x = a\sin u$	$a^2 - a^2\sin^2 u = a^2\cos^2 u$	$a\cos u$
$\sqrt{a^2+x^2}$	$x = a\tan u$	$a^2 + a^2\tan^2 u = a^2\sec^2 u$	$a\sec u$
$\sqrt{x^2-a^2}$	$x = a\sec u$	$a^2\sec^2 u - a^2 = a^2\tan^2 u$	$a\tan u$

Nice fact: there is very little choice for u.
$\therefore$ Make the u-sub. and see what happens ...

eg//
$$\int \frac{1}{(4-x^2)^{3/2}}\,dx$$

$$x = 2\sin u \to 4 - x^2 = 4\cos^2 u$$
$$dx = 2\cos u\,du$$

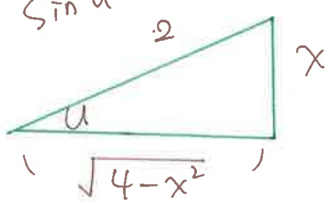
$$= \int \frac{2\cos u}{8\cos^3 u}\,du$$

$$= \frac{1}{4}\int \frac{1}{\cos^2 u}\,du = \frac{1}{4}\int \sec^2 u\,du$$

$$= \frac{1}{4}\tan u + C$$

$$= \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C \text{ //}$$

$\sin u = \frac{x}{2}$



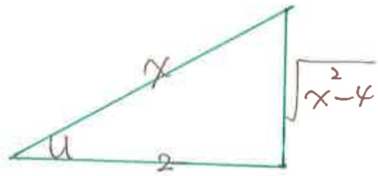
$\tan u = \frac{x}{\sqrt{4-x^2}}$

eg. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

$x = 2\sec u$; $dx = 2\sec u \tan u \, du$

$x^2-4 = (2\tan u)^2$

$\sec u = \frac{x}{2}$



$$= \int \frac{\cancel{2\sec u}\,\cancel{\tan u}}{4\sec^{\cancel{2}} u \cdot \cancel{2}\,\cancel{\tan u}}\, du$$

$$= \frac{1}{4}\int \frac{1}{\sec u}\, du = \frac{1}{4}\int \cos u \, du$$

$$= \frac{1}{4}\sin u + c$$

$$= \frac{1}{4}\,\frac{\sqrt{x^2-4}}{x} + c$$

eg. $\displaystyle\int \frac{x}{\sqrt{x^2+2x+5}}\, dx$

completing the square

$$= \int \frac{x}{\sqrt{(x+1)^2+4}}\, dx$$

$x+1 = 2\tan u$

$dx = 2\sec^2 u \, du$

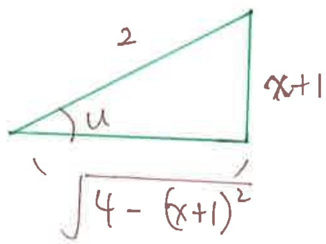
$(x+1)^2+4 = 4\sec^2 u$

$$= \int \frac{2\tan u - 1}{\cancel{2\sec u}} \cdot \cancel{2}\sec^{\cancel{2}} u \, du$$

$$= 2\int \sec u \tan u - \int \sec u \, du$$

$$= 2\sec u - \ln|\sec u + \tan u| + c$$

$$= 2\,\frac{2}{\sqrt{4-(x+1)^2}} - \ln\left|\frac{2}{\sqrt{4-(x+1)^2}} + \frac{x+1}{2}\right| + c$$



* Rational Functions

We evaluate $\int \frac{p(x)}{q(x)}\,dx$, where $p(x)$, $q(x)$ are polynomial, where $\deg p < \deg q$.

(otherwise, apply poly. division so that $\frac{p}{q} = r + \frac{p'}{q}$ w/ $\deg p' < \deg q$)

poly.

Fact from algebra

A real polynomial is called irreducible over $\mathbb{R}$ if it can not be written as product of real polynomials of lower degrees.

eg// x^2+1 is irreducible

x^2-4 is reducible since $x^2-4 = (x+2)(x-2)$

For $\deg q \le 3$, q is reducible if and only if it has a real root.

For q with degree >3, it's generally not easy to determine whether it's reducible or not.

FYI: (Einstein Criterion)

$q = a_n x^n + \cdots + a_0$; $a_i \in \mathbb{Z}$.

if there is a prime # p so that

$$\left\{\begin{array}{l} c \text{ divides } a_0, a_1, \ldots, a_{n-1} \\ c \text{ does not divide } a_n \\ c^2 \text{ does not divide } a_0 \end{array}\right. \Rightarrow q \text{ is irreducible.}$$

Eisenstein does not discuss ALL real polynomials.
and even if ⊛ fails, q might still be irreducible.

Lastly, any polynomial $q(x)$ can be written as

$$q(x) = q_1(x)^{r_1} \cdots q_m(x)^{r_m}$$

and each $q_i(x)$ is irreducible

* Partial Fraction Decomposition

Any rational function $\frac{P(x)}{q(x)}$ can be re-written as

$$\frac{P(x)}{q(x)} = \frac{P_{1,1}(x)}{q_1(x)} + \cdots + \frac{P_{1,r_1}(x)}{q_1(x)^{r_1}} + \cdots + \frac{P_{m,1}(x)}{q_m(x)} + \cdots + \frac{P_{m,r_m}(x)}{q_m(x)^{r_m}}$$

(**)

where $q(x) = q_1(x)^{r_1} \cdots q_m(x)^{r_m}$ is the prime decomposition of q.

and for each i, $\deg P_{i,j}(x) = \deg q_i - 1$

We hope that (RHS. of) (**) is easier (or possible) to integrate than $P(x)/q(x)$.

Of course, this decomposition is impracticle if $\deg q$ is too big.

eg. Evaluate $\int \frac{2x}{x^2-x-2}\,dx$

$$\frac{2x}{x^2-x-2} = \frac{2x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad \cdots (**)$$

Multiply $(x-2)(x+1)$ on both sides of $(**)$

$$2x = A(x+1) + B(x-2)$$
$$= (A+B)x + A-2B$$

comparing coefficients $\Rightarrow$ $\begin{cases} A+B=2 \\ A-2B=0 \end{cases} \longrightarrow \begin{cases} A=\frac{4}{3} \\ B=\frac{2}{3} \end{cases}$

$$\therefore \int \frac{2x}{x^2-x-2}\,dx = \frac{4}{3}\int \frac{1}{x-2}\,dx + \frac{2}{3}\int \frac{1}{x+1}\,dx$$
$$= \frac{4}{3}\ln|x-2| + \frac{2}{3}\ln|x+1| + C$$

Alternative way to solve $(**)$:

let $x=-1 \rightarrow -2 = -3B \rightarrow B = 2/3$

$x=2 \rightarrow 4 = 3A \rightarrow A = 4/3$

eg. Evaluate $\int \frac{dx}{x(x^2+x+1)}$

x^2+x+1 is irreducible since it has no real root

("b^2-4ac" $= 1-4 < 0$)

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$\rightsquigarrow \quad A(x^2+x+1) + (Bx+C)x = 1$$

$x=0$: $\quad A = 1$

$x=1$: $\quad 3 + B + C = 1$

$x=-1$: $\quad 1-(C-B) = 1$, where $1-(C-B) = 1+B-C$

$$\left.\begin{matrix} 3+B+C=1 \\ 1-(C-B)=1 \end{matrix}\right\} \quad \begin{matrix} B=-1 \\ C=-1 \end{matrix}$$

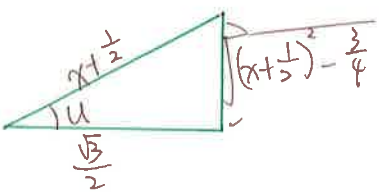
$$\int \frac{dx}{x(x^2+x+1)} = \int \frac{1}{x}\,dx - \int \frac{x+1}{x^2+x+1}$$

$$= \underbrace{\int \frac{1}{x}\,dx}_{\mathrm{I}} - \underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{\mathrm{II}} + \underbrace{\int \frac{x}{x^2+x+1}\,dx}_{\mathrm{III}}.$$

$\mathrm{I} = \ln|x|$

$\mathrm{II} = \ln|x^2+x+1|$

$$\mathrm{III} = \int \frac{x}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dx \overset{x+\frac{1}{2}=\frac{\sqrt{3}}{2}\sec u}{=} \int \frac{\frac{\sqrt{3}}{2}\sec u - \frac{1}{2}}{(\frac{\sqrt{3}}{2})^2\sec^2 u} \cdot \frac{\sqrt{3}}{2}\sec u \tan u\,du$$



$$= \int \tan u\,du - \frac{1}{\sqrt{3}}\int \frac{\tan u}{\sec u}\,du$$

$$= -\ln|\cos u| - \frac{1}{\sqrt{3}}\int \sin u\,du$$

$$= -\ln\left|\frac{\frac{\sqrt{3}}{2}}{x+\frac{1}{2}}\right| + \frac{1}{\sqrt{3}}\cos u$$

$$= -\ln\left|\frac{\sqrt{3}}{2x+1}\right| + \frac{1}{\cancel{\sqrt{3}}}\,\frac{\cancel{\sqrt{3}}}{2x+1} \quad //$$

eg. Evaluate $\int_1^3 \frac{x^2-4x+3}{x^3+2x^2+x}\,dx$

$$x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$$

$$\frac{x^2-4x+3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$\cdot\, x(x+1)^2$

$$A(x+1)^2 + Bx(x+1) + Cx = x^2-4x+3$$

$x=0 \rightarrow A = 3$

$x=-1 \rightarrow -C = 8 \rightarrow C = -8$

compare coefficient of x on both sides.

$$\underbrace{2A+B+C}_{B-2} = -4 \longrightarrow B = -2$$

$$\therefore \int_1^3 \frac{x^2-4x+3}{x^3+2x^2+x}\,dx = 3\ln|x|\Big|_1^3 - 2\ln|x+1|\Big|_1^3 - 8\int_1^3 \frac{1}{(x+1)^2}\,dx$$

$$\overset{u=x+1}{=} 3\ln 3 - 2\ln 4 - 8\int_2^4 \frac{1}{u^2}\,du$$

$$= \ln\frac{27}{16} + 8\cdot\frac{1}{u}\Big|_4^2$$

$$= \ln\frac{27}{16} + 8\left(\frac{1}{2}-\frac{1}{4}\right) = \ln\frac{27}{16} + 2$$

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