VIII. Transcendental Functions

Recall: $f: A \to B$, where A, B subsets of $\mathbb{R}$.

A is called the "domain" of f.

let $R(f) = \{ f(x) \in B \mid x \in A \}$, "range of f".

we may consider f as a function

$$f: A \to R(f).$$

Q: can we find a "inverse" of f? i.e.

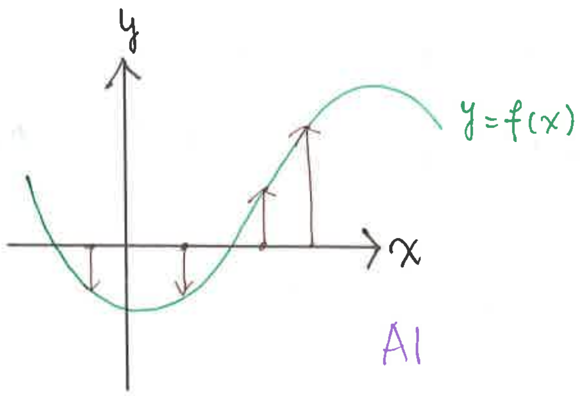
a function $g: R(f) \to A$ so that

$$g \circ f(x) = x \quad \text{for all } x \in A \ ? \quad \cdots ⊛$$

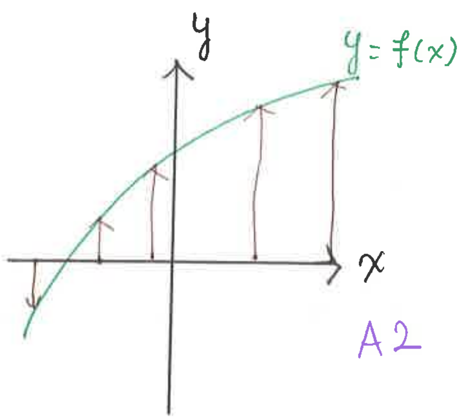
Such a function is called the "inverse" of f, written f^{-1}.

When does f^{-1} exist?

Intuitively, when we "graph" $y = f(x)$, we stand on an x and look "up or down" to locate the unique corresponding $f(x)$.

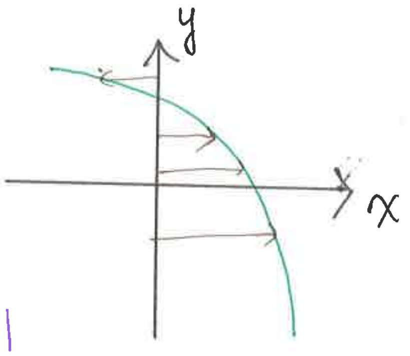


A1

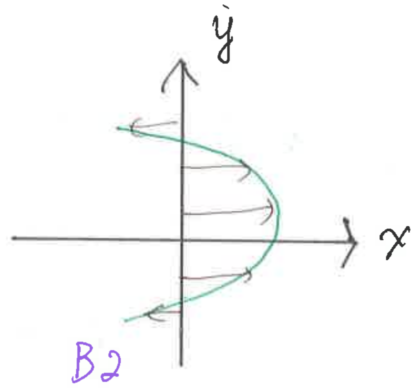


A2

For curve given as $x = g(y)$, we stand at some y and look for unique corresponding $g(y)$ value



B1



B2

∴ ⊛ is: start from x, looks up/down (f) → get a y value, and then, start from that y, look left or right (g), and get back the orginal x

observe, A2 is possible, A1 is not possible.

Reason: on A1, more than two x's mapped to the same y.

Defn (One-to-one Function)

A function is one-to-one, if no two distinct points is mapped to the same value.

ie.

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

eg,, $f : \mathbb{R} \to \mathbb{R}$ given by

$f(x) = 3x - 5$ is 1-1.

since $f(x_1) = f(x_2)$

$$3x_1 - 5 = 3x_2 - 5$$

$$\Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2.$$

eg,, $f : \mathbb{R} \to \mathbb{R}$ given by

$f(x) = x^2 + 1$ is not one-to-one

Since $f(x) = f(-x)$ for all x.

* Finding Inverses

- Let $y = f(x)$ and solve x as a function of y.

eg//

$f(x) = 3x - 5$

$y = 3x - 5$, $3x = y + 5 \Rightarrow x = \frac{1}{3}y + \frac{5}{3}$

$f^{-1}(y) = \frac{1}{3}y + \frac{5}{3}$

Check: $f^{-1}(f(x)) = f^{-1}(3x-5)$

$= \frac{1}{3}(3x-5) + \frac{5}{3} = x.$

eg//

$f(x) = (x+1)^3 + 2$

$y = (x+1)^3 + 2$

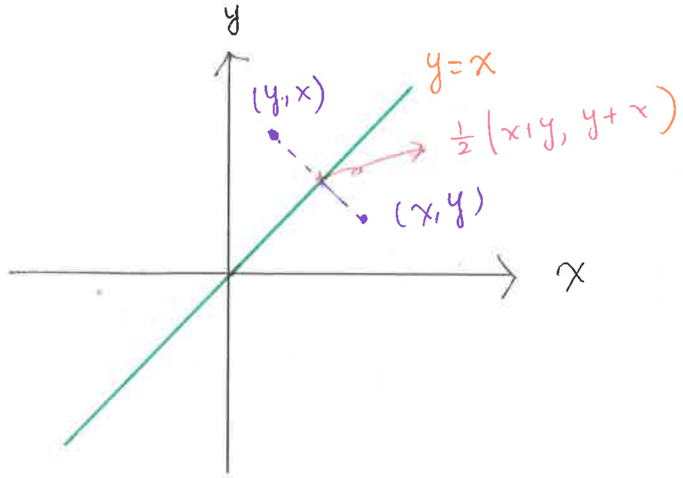
$(y-2)^{\frac{1}{3}} = x+1 \Rightarrow x = (y-2)^{\frac{1}{3}} - 1$

$f^{-1}(y) = (y-2)^{\frac{1}{3}} - 1$

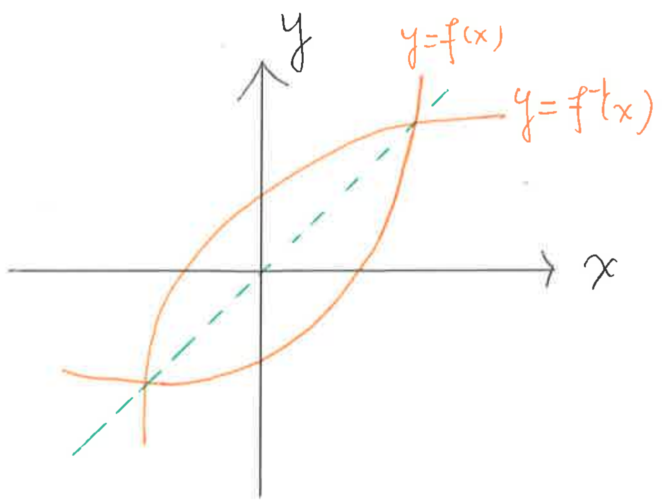
Graph of f^{-1}:

switching y & x. observe $y = f^{-1}(x)$ vs. $y = f(x)$.

For any point (x, y) on $\mathbb{R}^2$, (y, x) is its mirror image across the mirror $y = x$.



$\therefore$ The mirror image of $(x_0, f(x_0))$ across line $y = x$ is $(\underbrace{f(x_0)}_{x}, \underbrace{x_0}_{y})$ but $f^{-1}(\underbrace{f(x_0)}_{x}) = \underbrace{x_0}_{y}$ $\therefore$ $(f(x_0), x_0)$ lies on the graph $y = f^{-1}(x)$.



Thm,, If $f: I \to R(f)$ continuous, differentiable and one-to-one, then $f^{-1}: R(f) \to I$ cont. diff. 1-1.

$$f\,(f(x)) = x$$

$$\frac{d}{dx}\left[f^{-1}(f(x))\right] \underset{y=f(x)}{=} \left.\frac{df^{-1}}{dy}\right|_{f(x)} \frac{df}{dx}$$

$$\|$$

$$\frac{d}{dx}x = 1$$

$$\therefore \quad \left.\frac{df^{-1}}{dy}\right|_{f(x)}, \quad \frac{df}{dx} \neq 0 \quad \text{and.}$$

$$\left.\frac{df^{-1}}{dy}\right|_{f(x)} = \frac{1}{\frac{df}{dx}}$$

ie. At each point $a \in I$, $f(a) = b$, and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

eg,, Find $(f^{-1})'(\sqrt{3})$ for $f(x) = \tan x$; $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the only $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $f(a) = \tan a = \sqrt{3}$ is $\frac{\pi}{3}$, $f(\frac{\pi}{3}) = \sqrt{3}$

$$\therefore \quad (f^{-1})'(\sqrt{3}) = \frac{1}{f'(\frac{\pi}{3})} = \frac{1}{\sec^2\frac{\pi}{3}} = \left(\cos\frac{\pi}{3}\right)^2 = \frac{1}{4}$$

//

eg// Find a formula for $(f^{-1})'(x)$ when $f'(x) = f(x)$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$ since $f(f^{-1}(x)) = x$

$$= \frac{1}{f(f(x))} = \frac{1}{x}$$

* Logarithm Function

Q : What is the antiderivative of $\frac{1}{x} = x^{-1}$?

The rule $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ is not applicable since $n = -1$.

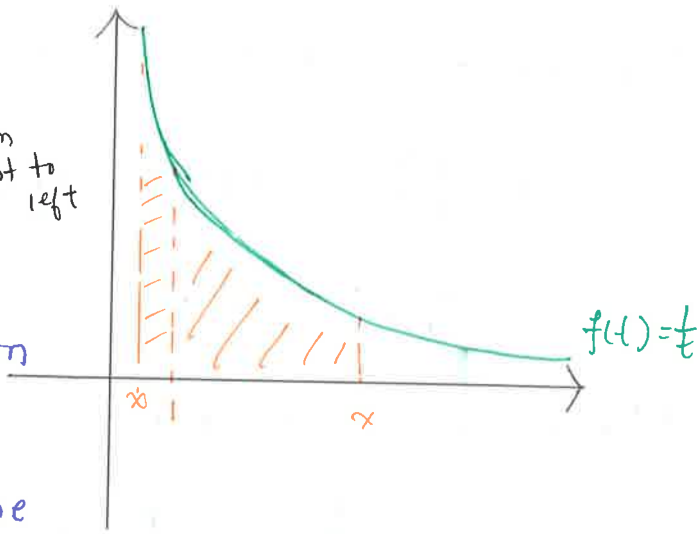
Let $L(x)$ be antiderivative of $\frac{1}{x}$.

$L'(x) = \frac{1}{x}$ or. $L(x) = \int_1^x \frac{1}{t} \, dt$

Some obvious facts

① Domain of L: $(0, \infty)$

② $L(x) \begin{cases} > 0 \; ; & x > 1 \\ = 0 \; ; & x = 1 \\ < 0 \; ; & x < 1 \end{cases}$ ← integrating from right to left



③ Range of $L = (-\infty, \infty) = \mathbb{R}$

④ $L'(x) = \frac{1}{x} > 0$ on its domain

$\therefore$ $L : (0, \infty) \to \mathbb{R}$ is one-to-one

(and L^{-1} exists)

Less obvious Facts :

Thm. For all $a, b > 0$, $L(ab) = L(a) + L(b)$

Pf. $\frac{d}{dx}[L(xb)] \underset{\text{chain rule}}{=} \frac{1}{xb} \cdot b = \frac{1}{x} = \frac{d}{dx}[L(x)]$

$$\therefore \frac{d}{dx}[L(xb) - L(x)] = 0$$

$$\rightsquigarrow L(xb) = L(x) + C.$$

$x = 1 \rightsquigarrow L(b) = L(1) + C = C.$

$$\therefore L(xb) = L(x) + L(b).$$

$x = a > 0 \rightsquigarrow L(ab) = L(a) + L(b).$ QED.

It follows that, for all $a, b > 0$,

$$L\left(\frac{1}{b}\right) + L(b) = L\left(\frac{1}{b} \cdot b\right) = L(1) = 0 \Rightarrow \underline{L\left(\frac{1}{b}\right) = -L(b)}.$$

$$\underline{L\left(\frac{a}{b}\right)} = L\left(a \cdot \frac{1}{b}\right) = L(a) + L\left(\frac{1}{b}\right) = \underline{L(a) - L(b)}$$

Moreover, we have

Thm. $L(a^{\frac{p}{q}}) = \frac{p}{q} L(a)$

$\forall \frac{p}{q} \in \mathbb{Q}$

Pf. claim : $\frac{d}{dx}[L(x^{\frac{p}{q}})] = \frac{d}{dx}[\frac{p}{q} L(x)]$

$$\frac{d}{dx}[L(x^{\frac{p}{q}})] = \frac{1}{\cancel{x^{p/q}}} \cdot \frac{p}{q} \cancel{x^{\frac{p}{q}-1}} = \frac{p}{q}\frac{1}{x} = \frac{d}{dx}\left[\frac{p}{q} L(x)\right] \checkmark$$

$$\therefore L(x^{\frac{p}{q}}) = \frac{p}{q} L(x) + C$$

$$x = 1 \rightsquigarrow \overset{0}{L(1)} = \frac{p}{q}\overset{0}{L(1)} + C \rightsquigarrow C = 0$$

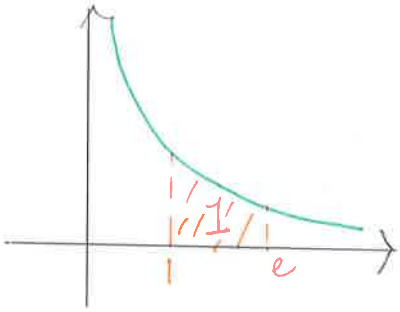
$\therefore L(x^{\frac{p}{q}}) = \frac{p}{q} L(x)$. let $x = a$. QED.

* e and ln

Since $L : (0, \infty) \to \mathbb{R}$ and $R(L) = \mathbb{R}$.

there is a number $e \in (0, \infty)$ s.t. $L(e) = \int_1^e \frac{1}{t}\,dt = 1$

$1 > 0 \Rightarrow e > 1$.



will show: $e \approx 2.71\cdots$

$\therefore \quad L(e^x) = x\,L(e) = x$

$\therefore \quad f(x) = e^x$ is the inverse function of L.

$\boxed{e^x \text{ satisfies all exponent rule } \; e^{a+b} = e^a \cdot e^b \, ; \; e^{-b} = \frac{1}{e^b} \, ; \; e^{a-b} = \frac{e^a}{e^b}}$

That is. $y = e^x \iff L(y) = x$.

Define (as expected) $L(x) := \ln x$ "natural log"

$$\ln(e^x) = x \quad \left(\text{and} \quad e^{\ln x} = x \right)$$

$\frac{d}{dx}$: $\quad \frac{1}{e^x}\left[\frac{d}{dx} e^x\right] = 1 \rightsquigarrow \frac{d}{dx} e^x = e^x \rightsquigarrow \frac{d^n}{dx^n} e^x = e^x$

The function $f(x) = e^x$ equals all its derivatives

to any order.

Such a function most suitably describes a quantity whose rate of of change depends on the quantity itself. eg: birth rate, population growth, radioactive growth/decay, bank statement balance.

* More on Logarithms *

Differentiation involving $\ln x$.

$x > 0$, $\quad \frac{d}{dx}(\ln x) = \frac{1}{x}$

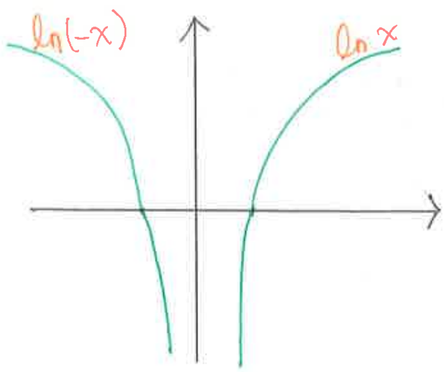
eg., $f(x) = \ln\left(x\sqrt{4+x^2}\right)$

$$f'(x) = \frac{\frac{d}{dx}\left[x\sqrt{4+x^2}\right]}{x\sqrt{4+x^2}} = \frac{\sqrt{4+x^2} + \frac{x^2}{\sqrt{4+x^2}}}{x\sqrt{4+x^2}}$$

$$= \frac{2x^2+4}{x(4+x^2)}$$

eg., $f(x) = \ln|x| : \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$$f(x) = \begin{cases} \ln x & ; \; x > 0 \\ \ln(-x) & ; \; x < 0 \end{cases}$$



$x > 0$, $\quad f'(x) = \frac{1}{x}$

$x < 0$, $\quad f'(x) = -\frac{1}{x}\cdot(-1) = \frac{1}{x}$

$\therefore \quad f'(x) = \frac{1}{x} \quad \forall \; x \neq 0.$

$\Rightarrow \quad \int \frac{1}{x}\,dx = \ln|x| + C \quad$ if x not necessarily > 0

Applications toward trig. integration:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx \qquad u = \cos x, \quad dx = \frac{du}{-\sin x}$$

$$= -\int \frac{\cancel{\sin x}}{u} \frac{du}{\cancel{\sin x}}$$

$$= -\ln|u| + c = -\ln|\cos x| + c$$

Similarly. $\int \cot x\, dx = \ln|\sin x| + c$

$$\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\cancel{\sec^2 x + \sec x \tan x}}{u} \frac{du}{\cancel{\sec^2 x + \sec x \tan x}}$$

$$= \ln|\sec x + \tan x| + c$$

Similarly

$$\int \csc x\, dx = \ln|\csc x - \cot x| + c$$

12/19 Ends

Simplifications on Product Rule.

$$g(x) = g_1(x) \cdots g_n(x)$$

$$\ln g(x) = \ln g_1(x) + \cdots + \ln g_n(x)$$

$$\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}$$

$$\Rightarrow g'(x) = g(x)\left[\sum_{i=1}^{n} \frac{g_i'(x)}{g_i(x)}\right]$$

eg.

$$g(x) = \frac{(x^2+1)^3(2x-5)^2}{(x^2+5)^2} = (x^2+1)^3(2x-5)^2(x^2+5)^{-2}$$

$$g'(x) = g(x)\left[\frac{3(x^2+1)^2\cdot 2x}{(x^2+1)^3} + \frac{2(2x-5)\cdot 2}{(2x-5)^2} - \frac{2(x^2+5)^{-3}\cdot 2x}{(x^2+5)^{-2}}\right]$$

$$= g(x)\left(\frac{6x}{x^2+1} + \frac{4}{2x-5} - \frac{4x}{x^2+5}\right).$$

//

* Calculus involving e^x

$$\frac{d}{dx}e^x = e^x \rightsquigarrow \frac{d^n}{dx^n}e^x = e^x$$

$$\int e^x\,dx = e^x + C.$$

eg// $$\frac{d}{dx}e^{\sqrt{x}+1} = e^{\sqrt{x}+1}\cdot\frac{d}{dx}(\sqrt{x}+1) = \frac{e^{\sqrt{x}+1}}{2\sqrt{x}}$$ //

eg// $$\frac{d}{dx}(e^{x^2}+1)^2 = 2(e^{x^2}+1)\frac{d}{dx}(e^{x^2}+1)$$
$$= 2(e^{x^2}+1)(e^{x^2}\cdot 2x)$$ //

eg// $$\frac{d}{dx}\left(e^{-2x}\cos x\right) = -e^{-2x}(-2)\cos x + e^{-2x}\sin x$$
$$= e^{-2x}(2\cos x + \sin x)$$ //

eg// $$\int x e^{x^2}\,dx$$
$$= \int \cancel{x} e^u \frac{du}{2\cancel{x}} \qquad u = x^2\ ;\ dx = \frac{du}{2x}$$
$$= \frac{1}{2}e^{x^2} + C.$$

eg// $$\int_0^{\ln 2}\frac{e^x}{e^x+1}\,dx \qquad u = e^x+1 \qquad dx = \frac{du}{e^x}$$
$$= \int_0^{\ln 2}\frac{\cancel{e^x}}{u}\frac{du}{\cancel{e^x}} = \ln|e^x+1|\Big|_0^{\ln 2} = \ln|e^{\ln 2}+1| - \cancel{\ln 1}$$
$$= \ln 3$$ //

Sketches of $\ln x$ and e^x.

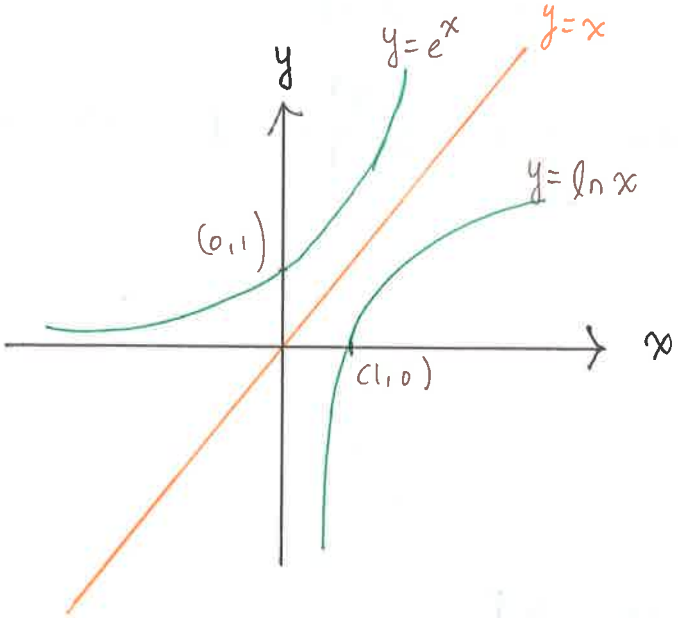
$$f(x) = \ln x = \int_1^x \frac{1}{t}\,dt \quad ; \quad x > 0$$

$f'(x) = \frac{1}{x} > 0 \Rightarrow f$ increasing.

$f''(x) = -\frac{1}{x^2} < 0 \Rightarrow$ graph always concaves down

and $f(1) = 0$

$\lim_{x\to\infty} f(x) = \infty$



and $f(x) = e^x$ is the inverse function to $\ln x$

$\therefore$ its graph is the mirror image of $\ln x$ across $y = x$.

Asymptotic Behaviors:

$$\lim_{x\to\infty} \ln x = \infty \rightsquigarrow \lim_{x\to\infty} e^x = \infty$$

$$\lim_{x\to 0^+} \ln x = -\infty \rightsquigarrow \lim_{x\to -\infty} e^x = 0.$$

Some standard curve sketching:

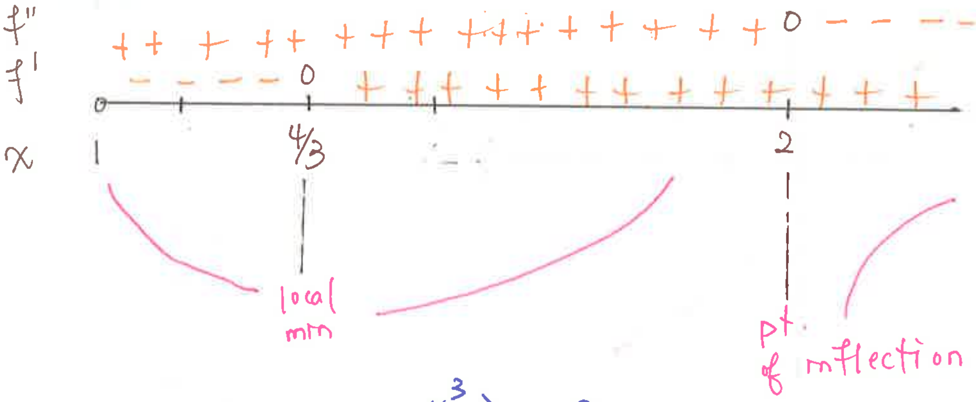
eg 11 $\quad f(x) = \ln\left(\frac{x^4}{x-1}\right) = \ln x^4 - \ln(x-1) = 4\ln x - \ln(x-1)$

$$f'(x) = \frac{3x-4}{x(x-1)} \quad ; \quad f''(x) = -\frac{(x-2)(3x-2)}{x^2(x-1)^2}$$

$= 0$ at $4/3$ $\qquad\qquad$ $= 0$ at $x = 2, \cancel{\tfrac{2}{3}}$ not in domain

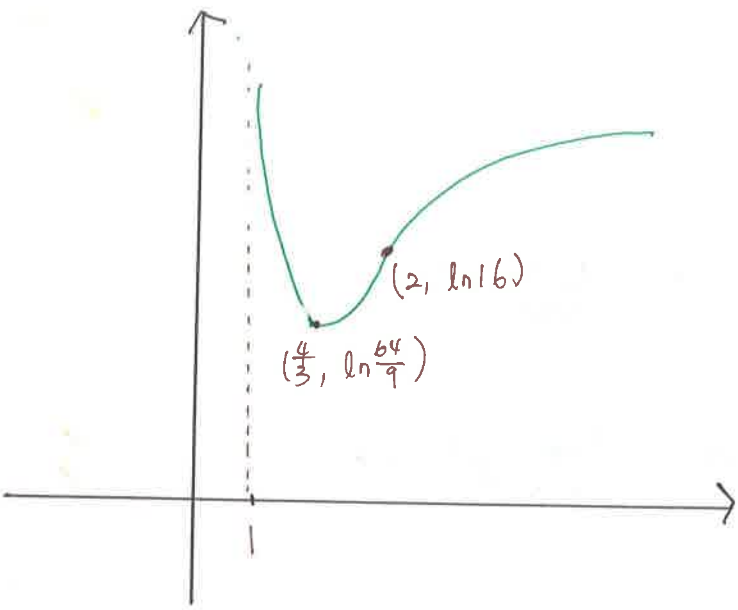
Domain: $\frac{x^4}{x-1} > 0$, i.e. $x > 1$.

Signs of Derivatives.



$f(\frac{4}{3}) = \ln\left(\frac{4^3}{3^2}\right) > 0$

$\therefore \quad f > 0 \quad \forall \quad x \in (1, \infty)$



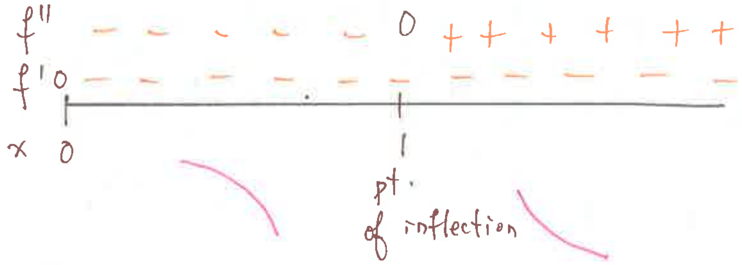
eg.11 $f(x) = e^{-\frac{x^2}{2}} : \mathbb{R} \to \mathbb{R}^+$

f is even function, $\therefore$ graph on $[0, \infty)$ only.

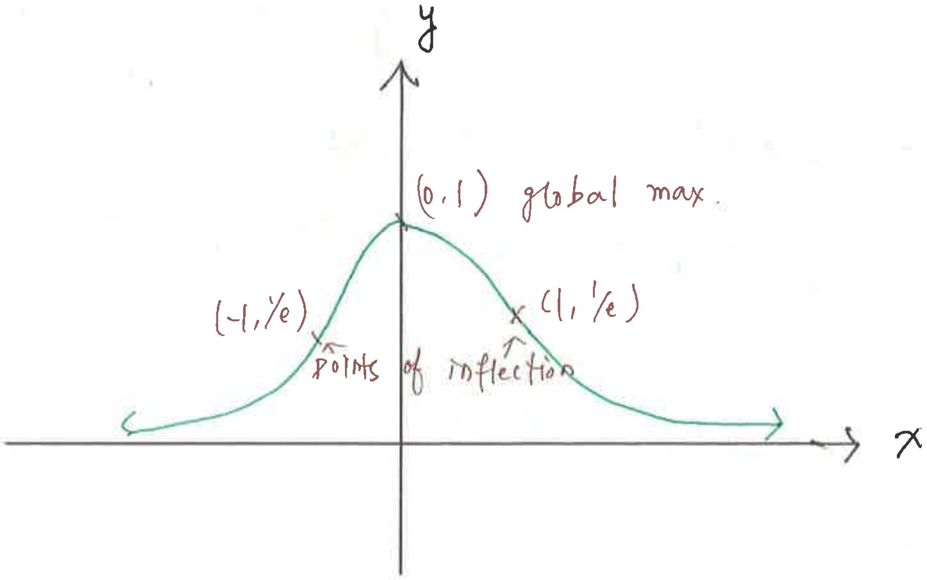
$$f'(x) = e^{-\frac{x^2}{2}} \cdot (-x) = -x e^{-\frac{x^2}{2}} \leq 0 \text{ on } [0, \infty)$$

$$f''(x) = -e^{-\frac{x^2}{2}} - x\left(-x e^{-\frac{x^2}{2}}\right) = (x^2 - 1) e^{-\frac{x^2}{2}}$$

$= 0$ at $x = 1$.



$\lim_{x \to \infty} f(x) = 0$



$\therefore$ abs. max at $x = 0$, where $f(x) = 1$

and $\lim_{x \to \infty} e^{-\frac{x^2}{2}} = 0$ as well eg. A rectangle has one side on x-axis and 2 upper vertices on the graph $y = f(x) = e^{-x^2}$. Find the upper two vertices so the area is maximum.

Locations of 4 vertices:

2 on x-axis : $(x_1, 0)$, $(x_2, 0)$

the other on graph : $(x_1, e^{-x_1^2})$, $(x_2, e^{-x_2^2})$

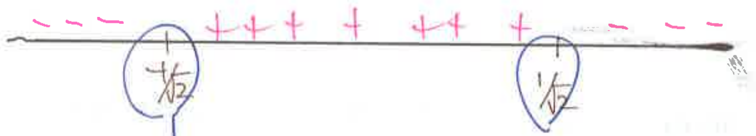
But $e^{-x_1^2}$ has to be $e^{-x_2^2}$ $\rightarrow$ $x_2 = -x_1$

$\therefore$ rectangle has vertices $(x, 0)$, $(-x, 0)$, (x, e^{-x^2}), $(-x, e^{-x^2})$

$$\therefore \quad A(x) = 2x\,e^{-x^2} \qquad (x \in \mathbb{R})$$

$$A'(x) = 2e^{-x^2} - 4x^2 e^{-x^2} = e^{-x^2}(2 - 4x^2)$$

$A' = 0$ at $x = \pm \frac{1}{\sqrt{2}}$



$$A\left(\pm \tfrac{1}{\sqrt{2}}\right) = \sqrt{2}\, e^{-1/2} > 0$$

and $\lim_{x \to \infty} A(x) = 0$

to study later.

$\therefore$ area is maximum at $x = \pm 1/\sqrt{2}$

* Arbitrary Powers. The Function $f(x) = \log_p x$

So far we have defined $f(x) = x^n$ for $n \in \mathbb{Q}$

we would like to extend the definition for $n \in \mathbb{R}$.

Note: for $n = \frac{p}{q} \in \mathbb{Q}$

$$e^{\ln\left(x^{\frac{p}{q}}\right)} = \frac{p}{q}$$

$$\parallel$$

$$e^{\frac{p}{q}\ln x} = e^{n \ln x}$$

makes sense for all $n \in \mathbb{R}$

$\therefore$ Define, $\forall\ n \in \mathbb{R}$.

$$x^n = e^{n \ln x}$$

Easy to verify: $\begin{cases} x^{r+s} = x^r \cdot x^s \\ x^{r-s} = x^r / x^s \\ (x^r)^s = x^{rs} \end{cases}$ for all $r, s \in \mathbb{R}$.

Differentiation formula continues to hold:

$$\frac{d}{dx}(x^r) = \frac{d}{dx}\left[e^{r\ln x}\right] = e^{r\ln x} \cdot \frac{d}{dx}(r \ln x)$$

$$= e^{r \ln x} \cdot \frac{r}{x} = r \cdot x^r \cdot \frac{1}{x}$$

$$= r \cdot x^{r-1}.$$

and therefore,

$$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C.$$

We may switch base and power: and consider

$$f(x) = p^x \quad ; \quad p > 0$$
$$= e^{x \ln p}$$

$$f'(x) = e^{x \ln p} \cdot \ln p = p^x \cdot \ln p$$

and $\therefore$ $$\int p^x \, dx = \frac{1}{\ln p} p^x + C$$

eg11 $$\int x \, 5^{-x^2} \, dx$$

$$= \int \cancel{x} \cdot \cancel{5^{-x^2}} \frac{du}{-2\ln 5 \cdot \cancel{x} \cdot \cancel{5^{-x^2}}}$$

$u = 5^{-x^2}$, $du = 5^{-x^2} \cdot \ln 5 \, (-2x) \, dx$

OR $dx = \dfrac{du}{-2(\ln 5) x \cdot 5^{-x^2}}$

$$= -\frac{1}{2\ln 5} u + C$$

$$= -\frac{5^{-x^2}}{2\ln 5} + C.$$

* Logarithm of Arbitrary Base

construct the inverse function to p^x, $p>0$

i.e. $f(p^x) = x$ ... ⊛

Note: $\ln p^x = x \ln p$ $\Rightarrow$ $\dfrac{\ln p^x}{\ln p} = x$

$\therefore$ $f(x) = \dfrac{\ln x}{\ln p}$ satisfies ⊛

and we define $f(x) := \log_p x$.

ie. $\log_p x = y \iff p^y = x$

then clearly, $\dfrac{d}{dx}(\log_p x) = \dfrac{1}{\ln p} \cdot \dfrac{1}{x}$

eg11 $\dfrac{d}{dx} \log_2 |3x+1|$

$$= \frac{1}{|3x+1|} \cdot \frac{1}{\ln 2} \cdot \frac{d}{dx}|3x+1|$$

$$= \begin{cases} \dfrac{3}{(\ln 2)|3x+1|} & ; \quad x > \frac{-1}{3} \\ \dfrac{-3}{(\ln 2)|3x+1|} & ; \quad x < \frac{1}{3} \end{cases}$$

function not differentiable at $x = -\frac{1}{3}$

* Exponential Growth / Decay

Exponential functions are standard models for quantities whose rate of change depend on itself.

A primitive example:

$$f'(t) = k f(t) \;: \quad t \text{ in some interval}$$

$$f'(t) - k f(t) = 0$$

$\cdot e^{-kt}$:

$$e^{-kt} f'(t) - k e^{-kt} f(t) = 0$$

$$\frac{d}{dt}\left[e^{-kt} f(t)\right]$$

$$\therefore \quad e^{-kt} f(t) = C \Rightarrow \underline{f(t) = Ce^{kt}}$$

$t = 0$:

$$f(0) = C \Rightarrow f(t) = f(0)e^{kt}$$

$k > 0 \rightarrow$ exponential growth

$k < 0 \rightarrow$ " decay.

eg// suppose the population of the world is modelled by exponential function., and given

$t=0 \rightarrow P(1980) = 4.5$ (Billion)

$t=20 \rightarrow P(2000) = 6$ (Billion) $\Rightarrow P(2010) = ?$

$$P(t) = P(0)e^{kt} = 4.5e^{kt}$$

$$P(20) = 4.5e^{20k} = 6 \Rightarrow k = \frac{1}{20}\ln\frac{6}{4.5} \simeq 0.0143$$

Year 2010, $t = 30$

$$P(30) = 4.5e^{20 \cdot 0.0143} \simeq 6.164 \text{ (Billion)}$$

eg4 (Half-Life)

$A(t)$ = amount of radioactive substance.

the half life T is the time it takes s.t.

$$\left.\begin{array}{l} A(T) = \frac{1}{2}A(0) \\ \quad \| \\ A(0)e^{kT} \end{array}\right\} \Rightarrow T = -\frac{1}{k}\ln 2.$$

if if a subspace has half life of 8 years, $A(16) = ?$

$$T = -\frac{1}{k}\ln 2 = 8$$

$$\Rightarrow k = \frac{-\ln 2}{8}$$

$$\therefore A(16) = A(0)e^{16k} = A(0)e^{2\cdot 8k}$$

$$= A(0)e^{-2\ln 2} = A(0)e^{\ln\frac{1}{4}}$$

$$= A(0)\cdot\frac{1}{4}$$

$\therefore$ $\frac{1}{4}$ of the original amount remained.

Compound Interest.

$A(t) =$ Balance at time t.

$r =$ annual interest.

$A_0 = A(0)$. initial deposit.

Let $n = \#$ of times the interest is compounded. (paid)

($n=1 \rightarrow$ annually, $n=4 \rightarrow$ quarterly, $n=12$, monthly, ...)

$n=1$ $A_n(1) = A_0(1+r)$

$n=4$ $A_4(1) = A_0(1+\frac{r}{4})(1+\frac{r}{4})(1+\frac{r}{4})(1+\frac{r}{4}) = A_0(1+\frac{r}{4})^4$

$n=12$ $A_{12}(1) = A_0(1+\frac{r}{12})^{12}$

For general n, $A_n(1) = A_0(1+\frac{r}{n})^n$.

If the interest is compounded continuously, $n \rightarrow \infty$.

$$A_\infty(1) = A_0 \cdot \lim_{n\rightarrow\infty}(1+\frac{r}{n})^n \quad \cdots ①$$

On the other hand, we have.

$$r\frac{1}{n}A(t) \le \underbrace{A(t+\tfrac{1}{n}) - A(t)}_{\substack{\text{interest earned}\\ \text{b/w } t \text{ and } t+}} \le r\cdot\frac{1}{n}A(t+\tfrac{1}{n})$$

$$rA(t) \le \frac{A(t+\frac{1}{n}) - A(t)}{\frac{1}{n}} \le rA(t+\tfrac{1}{n})$$

$n\rightarrow\infty$ $\quad rA(t) \le A'(t) \le rA(t) \Rightarrow A'(t) = rA(t)$

$\Rightarrow A(t) = A_0e^{rt} \Rightarrow A(1) = A_0e^r \quad \cdots ②$

CONT)

① , ② $\rightarrow$ $$e^r = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$$

$r=1 \rightarrow$ $$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \simeq 2.71 \ldots$$

eg. Find the annual interest rate needed if \$6000 will grow into \$10,000 in 8 years, with interest compounded continuously ...

$$10000 = 6000\, e^{8r}$$

$$\Rightarrow \frac{1}{8}\ln\frac{10}{6} = r \quad \rightarrow \quad r \simeq 0.064 = 6.4\%$$

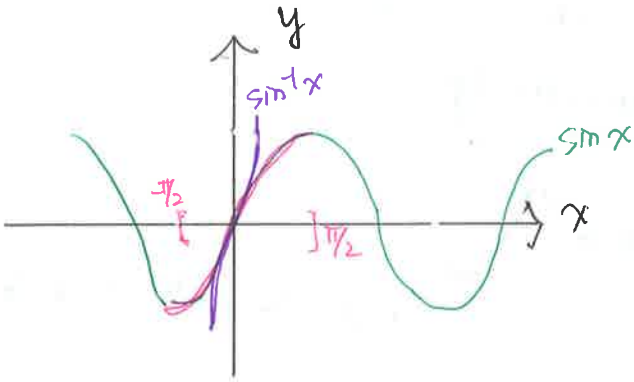
impossible in real life!!

(continous interest is also rarely seen in real banking)

* Inverse Trigonometric Functions

Trigonometric functions are one-to-one if domains are appropriately chosen :

sin, csc, tan :



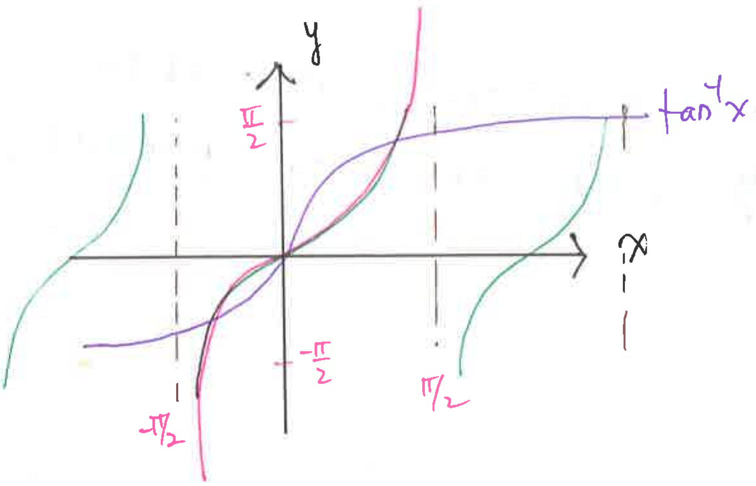
$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

is 1-1

$\therefore$ May define

$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

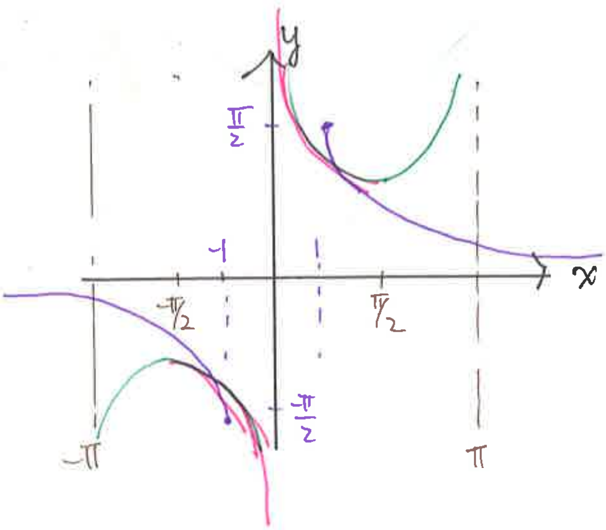
s.t. $\sin^{-1}(\sin x) = x$
$\qquad = \sin(\sin^{-1} x)$



$\tan : [-\frac{\pi}{2}, \frac{\pi}{2}] \to (-\infty, \infty)$ is 1-1

$\therefore \tan^{-1} : (-\infty, \infty) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

is defined



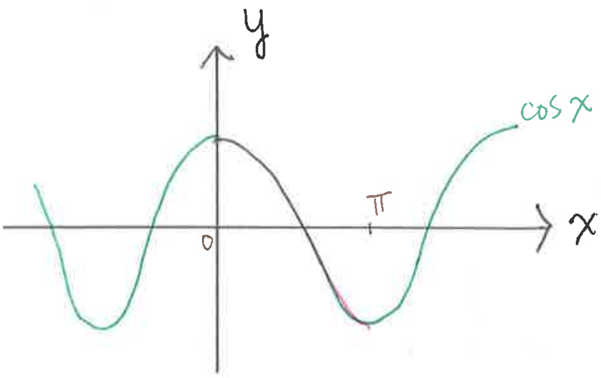
$\csc \quad [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} \to \mathbb{R} \setminus (-1, 1)$

is 1-1

$\therefore$ May define

$\csc^{-1} : \mathbb{R} \setminus (-1, 1) \to [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

cos, sec, cot.



$\cos : [0, \pi] \to [-1, 1]$ is 1-1

$\therefore \cos^{-1} : [-1, 1] \to [0, \pi]$ is defined.

Following similar reasons,

$\cot^{-1} ; [0, \pi] \to (-\infty, \infty)$, $\sec^{-1} : [0, \pi] \setminus \{\frac{\pi}{2}\} \to \mathbb{R} \setminus (-1,1)$

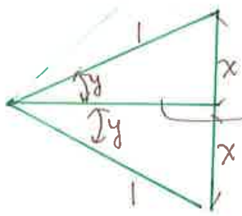
are defined.

Calculus of Inverse trig:

$y = \sin^{-1} x \rightsquigarrow \sin y = x$

$\frac{d}{dx} : \cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

But $\sin y = x$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

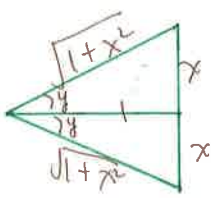


$\to \cos y = \sqrt{1 - x^2}$

$\therefore \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$

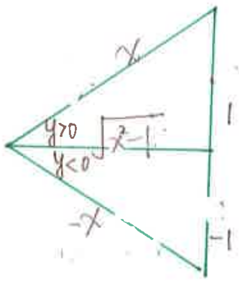
and $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C$

$y = \tan^{-1} x \longrightarrow \tan y = x \rightsquigarrow \frac{dy}{dx} \sec^2 y = 1$



$\rightsquigarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{1 + x^2}$

$y = \csc^{-1} x$; $\csc y = x \longrightarrow$ $x > 0 \Leftrightarrow y > 0$
$\quad (<) \quad\quad (<)$



$$(-\csc y \cot y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\csc y \cot y}$$

$$= \begin{cases} -\dfrac{1}{x\sqrt{x^2-1}} & ; \ x > 0 \\ -\dfrac{1}{-x\sqrt{x^2-1}} & ; \ x < 0 \end{cases}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$

Similarly,

$$\begin{cases} \dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \\ \dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2} \\ \dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2-1}} \end{cases}$$