VI. Vectors in $\mathbb{R}^n$ (Basic)

Q: Is $\mathbb{R}^3$ enough ?

Q: What do "distance, angle, length" mean in general $\mathbb{R}^n$? $(n > 3)$

Q: Points vs. Vectors ?

*Intuitions vs. Abstract Definitions.

Intuitions: direct constructions from what we see / touch.

Abstract Definitions: mathematical framework inspired by intuitions, constructed by logics (and some $\underbrace{\text{undefined building blocks}}_{\text{axioms.}}$)

Linear algebra is the first (easiest) generalization of intuitive spaces $(\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3)$ into abstract framework.

* Points vs. Vectors.

$n=3$. Traditionally,

$$\underbrace{\mathbb{R}^3}_{D_1} = \{ \underbrace{(x, y, z)}_{\text{point}} \mid x, y, z \in \mathbb{R} \}$$

Vectors in $\mathbb{R}^3$

$$\underbrace{\vec{\mathbb{R}^3}}_{D_2} = \{ \langle x, y, z \rangle \mid \langle x, y, z\rangle \text{ is an arrow such that, when placed at } (p,q,r) \in \mathbb{R}^3, \text{ ended at } (p+x, q+y, r+z) \in \mathbb{R}^3 \}$$

We define, on $\mathbb{R}^3$

- scalor multiplication

$$\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$$

$$(\alpha, (x,y,z)) := \alpha \cdot (x, y, z) := (\alpha x, \alpha y, \alpha z)$$

- Addition

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

O_1

Similarly, on $\vec{\mathbb{R}^3}$

- scalor multiplication

$$\mathbb{R} \times \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3}$$

$(\alpha, \langle x, y, z\rangle) \to \langle \alpha x, \alpha y, \alpha z\rangle$ = an arrow when placed at (p, q, r), ends at $(p+\alpha x, q + \alpha y, r + \alpha z)$

- Addition

$$\vec{\mathbb{R}^3} \times \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3}$$

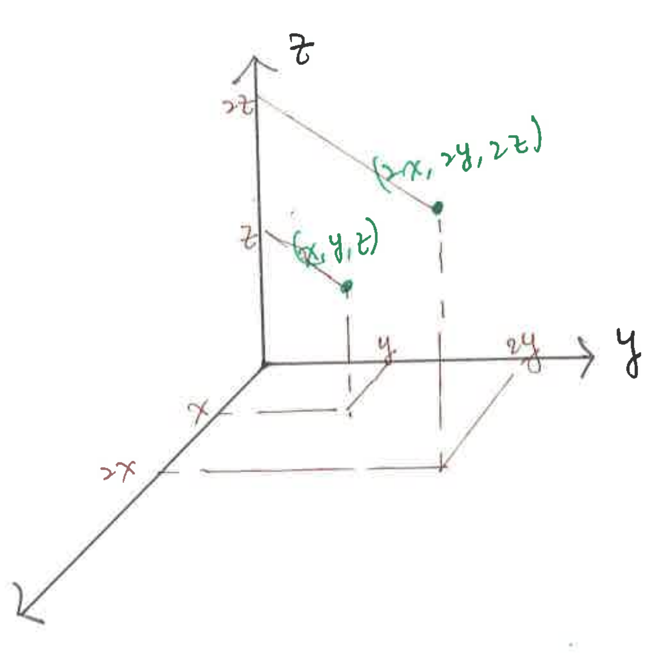
$$\langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$$

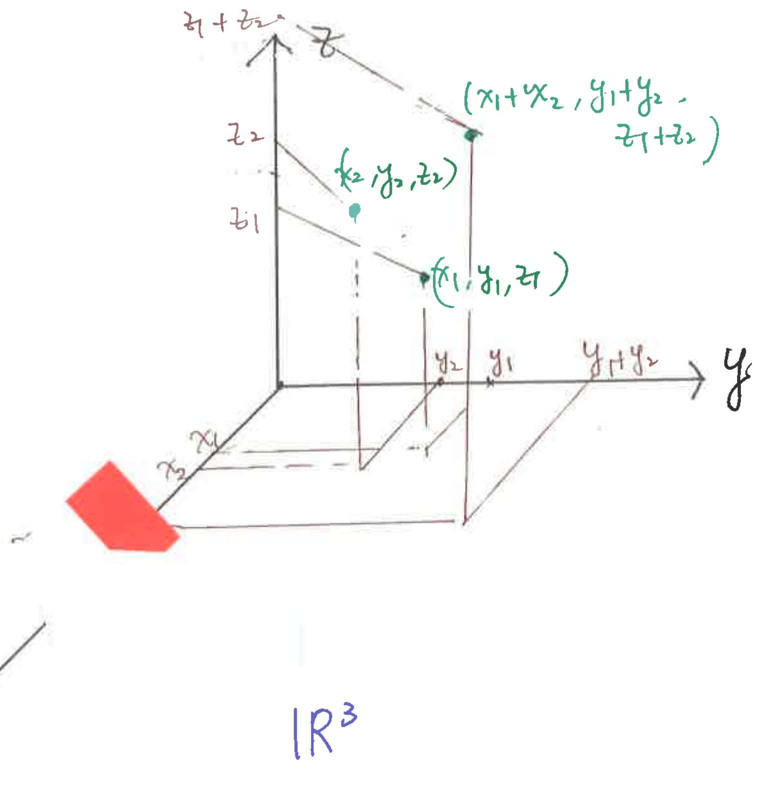
$= \ \because \ (p + x_1 + x_2, q + y_1 + y_2, r + z_1 + z_2)$

O_2

Geometric Representation

G1

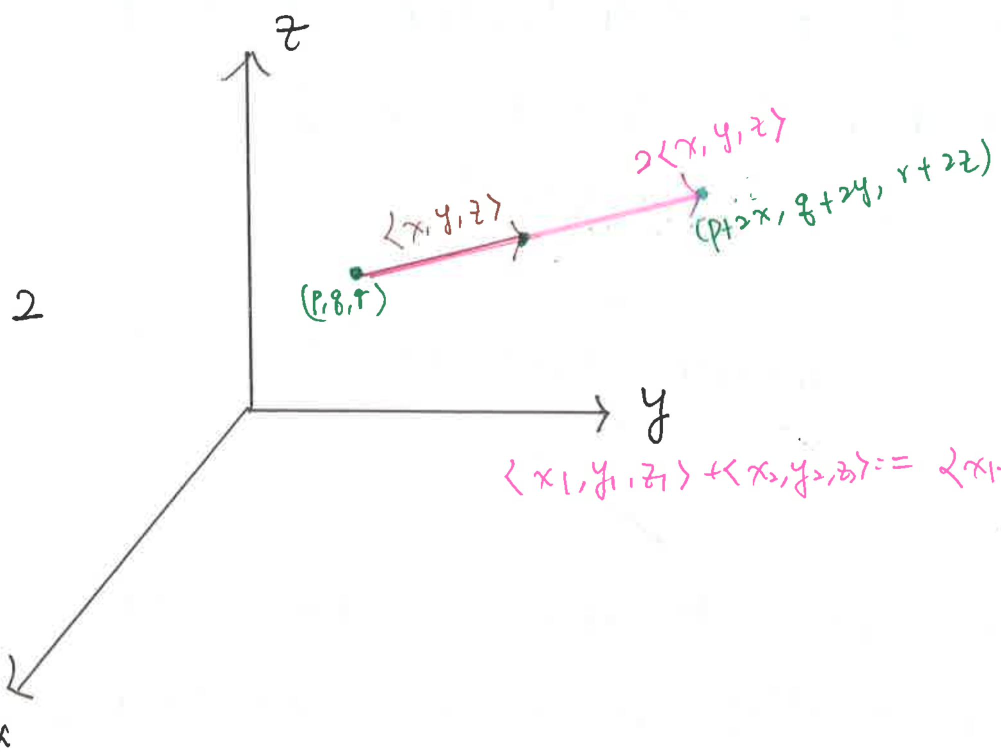




$\mathbb{R}^3$

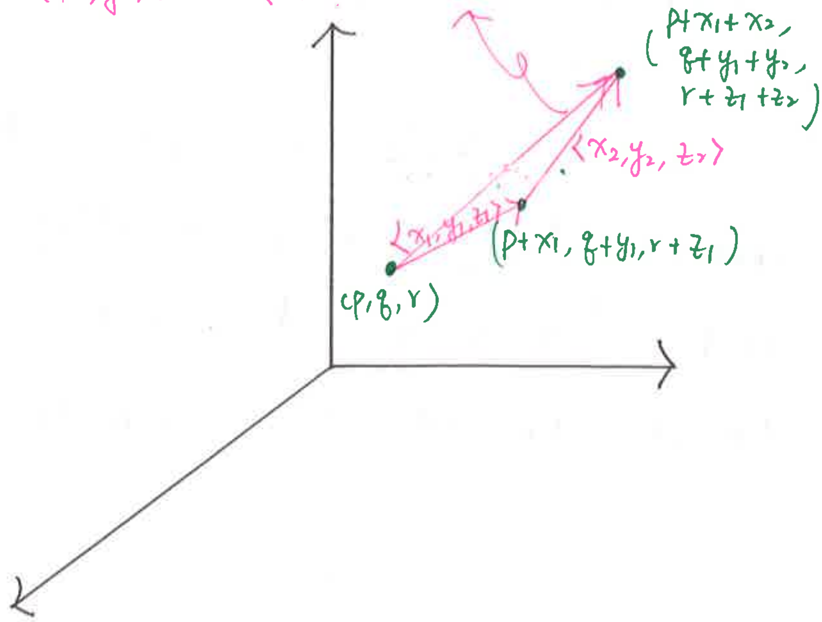
Pick any $(p,q,r) \in \mathbb{R}^3$

G2



$\vec{\mathbb{R}}^3$

$\langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle := \langle x_1+x_2, y_1+y_2, z_1+z_2\rangle$



$\vec{\mathbb{R}}^3 \simeq \mathbb{R}^3$

$\langle x, y, z\rangle \leftrightsquigarrow (x, y, z)$,

A vector $\langle x, y, z\rangle$ defines a point (x, y, z) by placing it at $(0,0,0)$ (clearly 1-1)

A point (x, y, z) defines a vector (arrow) $\langle x, y, z\rangle$ by connecting from $(0,0,0)$ to it. (then we know where it ends if placed at general (p, q, r)).

Geometrically, G1 coincides w/ G2 if we let $(p, q, r) = (0,0,0)$ & $\langle x_2, y_2, z_2\rangle$ starting from $(0,0,0)$ (Then $\langle \cdots \rangle = (\cdots)$)

& $D_1 = D_2$, $O_1 = O_2$ as well.

* We make NO distinction b/w vectors and points.

Linear algebra studies all objects that are formed by scalar multiplication & addition, and maps between them that preserve these operations (linear maps)

G_1 & G_2 no longer available for $\mathbb{R}^n$ with $n > 3$.
but D & O can be easily generalized.

We define the "(real) vector space" of dimension n to be $(\mathbb{R}^n, \cdot, +)$ consisted of

set $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ and operations

$$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$
$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(x_1, \dots, x_n) + (x_1', \dots, x_n') = (x_1 + x_1', \dots, x_n + x_n')$$

& write $0 = (0, \dots, 0)$ when no confusion arises.

1 10/29 Planned Endpoint 1

* Lines.

an object uniquely determined by two points.

(Note: a polynomial defining equation $P(x_1, \dots, x_n)$ of degree 1 is not a line when $n > 2$. $\therefore$ not appropriate for generalized definitions for lines)

Given $p \overset{\neq 0}{\in} \mathbb{R}^n$

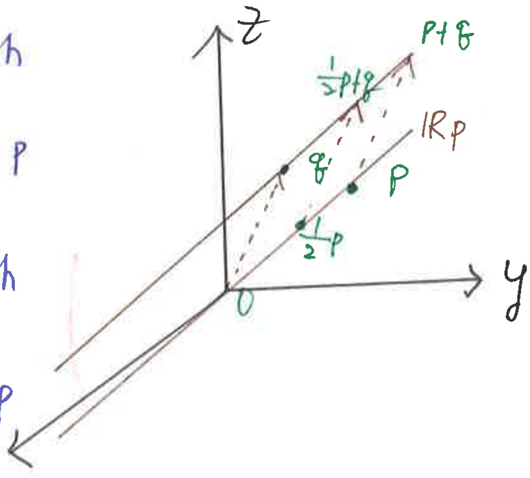
$\mathbb{R}p = \{t \cdot p \mid t \in \mathbb{R}\}$; line through origin in direction of p

(clearly, $0 \in \mathbb{R}p$)

For $q \in \mathbb{R}^n$

$q + \mathbb{R}p = \{q + tp \mid t \in \mathbb{R}\}$; line through q in direction of p

Note: $p \notin q + \mathbb{R}p$ if $q \neq 0$ }



ie a line determined by p & q doesn't contain p.

More explicitely,

If $p = (p_1, \dots, p_n)$ & $q = (q_1, \dots, q_n)$

$$p + \mathbb{R}p = \{(q_1 + (p_1 - q_1)t, \dots, q_n + (p_n - q_n)t) \mid t \in \mathbb{R}\}$$

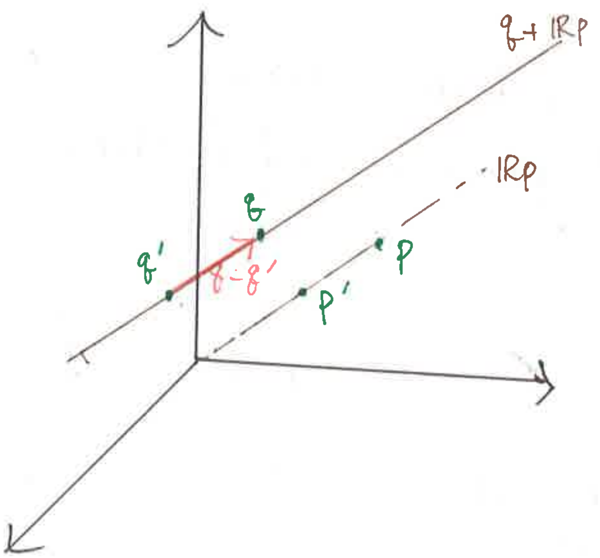
'parametric representation of a line'

Of course, choice of $\underbrace{p \text{ \& } q}_{\text{for the same line}}$ is not unique

(eg. may replace p by $2p$)

When are $p + \mathbb{R}p = p' + \mathbb{R}p'$?

$n = 3$:



Need p, p' both on $\mathbb{R}p$

$\Longleftrightarrow$ q, q' " " $q + \mathbb{R}p$

$\Updownarrow$

$p' = t_1 p$ for some $t_1 \in \mathbb{R}$, $t_1 \neq 0$

$q' \in q + \mathbb{R}p$

$\Longleftrightarrow$ $q' = q + t_2 p$ OR

$q' - q = t_2 p$ for some $t_2 \in \mathbb{R}$

$\Longleftrightarrow$ $q' - q \in \mathbb{R}p$

Prop. Let $L = q + \mathbb{R}p$ & $L' = q' + \mathbb{R}p'$ then

$$L = L' \iff \mathbb{R}p = \mathbb{R}p' \text{ and } q - q' \in \mathbb{R}p \ (= \mathbb{R}p')$$

pf. $\Rightarrow$ Given $L = L'$. ⊛ let $q \in L$ & $q' \in L'$

⊛ $\Rightarrow q \in L'$ & $q' \in L$.

$L = L' \Rightarrow \begin{cases} q' \in L \Rightarrow \\ q \in L' \Rightarrow \end{cases} \begin{cases} q' = q + \alpha p \\ q = q' + \beta p' \end{cases} \Rightarrow \begin{cases} q - q' = -\alpha p \ \cdots ① \\ q - q' = \beta p' \ \cdots ② \end{cases} \Rightarrow q - q' \in \mathbb{R}p.$ ✓

$\mathbb{R}p = \mathbb{R}p'$: (Need $p = \lambda p'$ for some $\lambda \neq 0$)

If $q \neq q'$; $\alpha, \beta \neq 0 \Rightarrow p = -\frac{\beta}{\alpha}p' \Rightarrow \mathbb{R}p = \mathbb{R}p'$ (Check)

If $q = q'$, $q + p \in \overset{L}{\overset{\shortparallel}{q + \mathbb{R}p}} = \overset{L'}{\overset{\shortparallel}{q' + \mathbb{R}p'}} = q + \mathbb{R}p'$
$= \{q + tp' \mid t \in \mathbb{R}\}$

$\therefore \cancel{q} + p = \cancel{q} + \lambda p'$ for some $\overset{\neq 0}{\lambda} \in \mathbb{R}$ (or $p = 0$)

$\Rightarrow \mathbb{R}p = \mathbb{R}\cdot p'$

$\Leftarrow$ Given $q - q' = \alpha p$ & $p = \beta p'$ $(\therefore \beta \neq 0)$

Show $L = q + \mathbb{R}p = L' = q' + \mathbb{R}p'$

$\subseteq$

Take $q + \lambda p \in L$? $q + \overset{\in L}{\lambda p} = q' + \alpha p + \lambda p = q' + \alpha\beta p' + \lambda\beta p'$
$= q' + (\alpha\beta + \lambda\beta)p' \in L'$

$\supseteq$

Take $q' + \mu p' \in L'$? $q' + \overset{\in L'}{\mu p'} = q - \alpha p + \frac{\mu}{\beta}p = q + \left(\frac{\mu}{\beta} - \alpha\right)p \in L.$

$\boxed{}$

Scalor multiplication & addition on $\mathbb{R}^n$ satisfy:

V1 $p+q = q+p \quad \forall p, q \in \mathbb{R}^n$

V2. $(p+q)+r = p+(q+r) \quad \forall p, q, r \in \mathbb{R}^n$

V3. $\exists\, 0 \in \mathbb{R}^n$ s.t. $p+0 = p \quad \forall p \in \mathbb{R}^n$

V4. $\forall p\ \mathbb{R}^n$, $\exists\, q \in \mathbb{R}^n$ s.t. $p+q=0$ (write $q =$ "$-p$")

V5. $(\alpha\beta)\cdot p = \alpha(\beta\cdot p) \quad \forall \alpha, \beta \in \mathbb{R}$ & $p \in \mathbb{R}^n$

V6. $\alpha\cdot(p+q) = \alpha\cdot p + \alpha\cdot q \quad \forall \alpha \in \mathbb{R}$ & $p, q \in \mathbb{R}^n$

V7. $(\alpha +_{\mathbb{R}} \beta)\cdot p = \alpha\cdot p + \beta\cdot p \quad \forall \alpha, \beta \in \mathbb{R}$ & $p \in \mathbb{R}^n$

V8. $1\cdot p = p \quad \forall p \in \mathbb{R}^n$.

V1 ~ V8 are trivial with $+, \cdot$ defined above. But not so if the space is not $\mathbb{R}^n$ $+, \cdot$ are not defined as we do. (Abstract Vector Space)

$(\mathbb{R}^n, +, \cdot)$ may be equipped with more structures: (operations)

For instance: Inner Product.
(not required for a vector space)

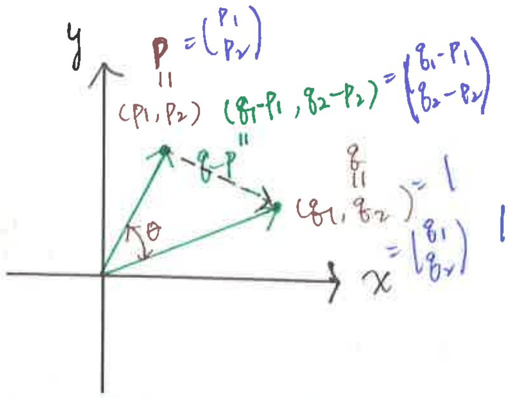
Change of notation from now on

$$(a_1, \ldots, a_n) \in \mathbb{R}^n$$
$$\Updownarrow$$
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$
$$\left(= (a_1, \ldots, a_n)^T \right)$$

(so we can apply matrices)

* Inner Product

↳ A measure on how parallel two lines are, and therefore give rise to the concept of angles.



$n = 2$

law of cosine:

$$\|p\|\|q\|\cos\theta = \frac{\|p\|^2 + \|q\|^2 - \|q - p\|^2}{2}$$

$$= \frac{p_1^2 + p_2^2 + q_1^2 + q_2^2 - (q_1 - p_1)^2 - (q_2 - p_2)^2}{2}$$

$$= p_1 q_1 + p_2 q_2 := p \cdot q \quad (\text{OR } \langle p, q \rangle)$$

$p \perp q \Rightarrow \mathbb{R}p$ & $\mathbb{R}q$ are "least parallel" $\Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow p \cdot q = 0$

" $p \cdot q$ " is something we can generalize to $\mathbb{R}^n$:

Def'n $p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$, $q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{R}^n$,

$$p \cdot q := \sum_{i=1}^{n} p_i q_i$$

and define angle between $\mathbb{R}p$ & $\mathbb{R}q$

$$\theta = \cos^{-1}\left(\frac{p \cdot q}{\|p\|\,\|q\|}\right).$$

Two points (vectors) are called orthogonal if

$$p \cdot q = 0.$$

Observe:

$$\begin{cases} (p+q)\cdot r = p\cdot r + q \cdot r & \forall\, p, q, r \in \mathbb{R}^n \\ (p\cdot q = q\cdot p) & \\ \alpha p \cdot q = \alpha (p\cdot q) & \forall\, \alpha \in \mathbb{R};\ p, q \in \mathbb{R}^n \\ p\cdot p \geq 0 & \forall\, p \in \mathbb{R}^n \end{cases}$$

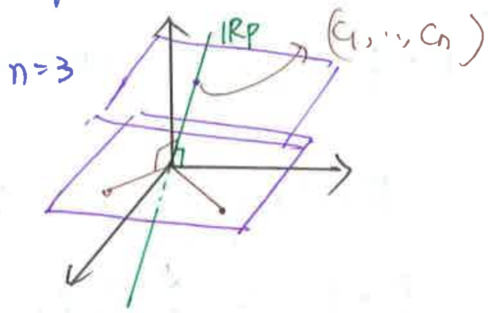
Any operation $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying ↑ is called an inner product. & $p\cdot q = \sum p_i q_i$ is called

Inner product also gives rise to hyper planes, objects complementary to lines.

Def.11

A hyperplane with normal vector p is

$$H := \{ x \in \mathbb{R}^n \mid x \cdot p = 0 \}$$



if $p = (p_1, \dots, p_n)^T$

$$H := \{ (x_1, \dots, x_n)^T \in \mathbb{R}^n \mid p_1 x_1 + \dots + p_n x_n = 0 \}$$

Shifting H by $(c_1, \dots, c_n)^T$, we have affine hyperplane

$$H_c = \{ (x_1, \dots, x_n)^T \in \mathbb{R}^n \mid p_1 x_1 + \dots + p_n x_n = c \}$$

where $c = -\sum_{i=1}^{n} c_i p_i$

Note:

$$x, y \in H \Rightarrow (x+y)\cdot p = x\cdot p + y\cdot q = 0+0=0$$
$$(\alpha x)\cdot p = \alpha(x\cdot p) = \alpha\cdot 0 = 0$$
$$\therefore \quad x+y,\ \alpha x \in H. \quad \& \ 0 \in H$$

$$x, y \in H_c \text{ for } c\neq 0 \Rightarrow (x+y)\cdot p = x\cdot p + y\cdot q = 2c \neq c$$
$$(\alpha x)\cdot p = \alpha c \neq c \text{ if } c\neq 0$$
$$\therefore \quad x+y,\ \alpha x \notin H_c \quad \& \ 0 \notin H_c$$

$(+,\cdot)$ is "H invariant" but not "H_c invariant".

That is, $(H,+,\cdot)$ is itself a vector space.

Def. Given $(\mathbb{R}^n,+,\cdot)$, a subset $W \subseteq \mathbb{R}^n$ is called a subspace of $(\mathbb{R}^n,+,\cdot)$ if ...

- $0 \in W$
- $x, y \in W \Rightarrow x+y \in W$
- $\alpha\cdot x \in W \ \forall \alpha \in \mathbb{R},\ x \in W$.

$(H,+,\cdot)$ is a subspace of $(\mathbb{R}^n,+,\cdot)$ but $(H_c,+,\cdot)$ $c\neq 0$ is not.

(Non-)Example of subspaces:

- $(\mathbb{R}^n,+,\cdot)$ is a subspace of itself
- $\{0\} \subseteq \mathbb{R}^n$ is a subspace
- $\mathbb{R}p$ " " "
- $\{(0, x_2, \ldots, x_n)\}$ is a subspace
- $q+\mathbb{R}p$ is not subspace. if $q \notin \mathbb{R}p$

Generating elements in a subspace:

if $v_1, \ldots, v_m \in \underset{\text{subspace}}{W}$,

$$\underbrace{\alpha_1 v_1 + \cdots + \alpha_m v_m}_{\text{linear combination}} \in W \quad \text{(use induction)}$$

$$Sp(v_1, \ldots, v_m) := \{\alpha_1 v_1 + \cdots + \alpha_m v_m \mid \alpha_i \in \mathbb{R}\} \subseteq W$$

"span"

Any subspace that contains $v_1, \ldots, v_m$ also contains $Sp(v_1, \ldots, v_n)$

Prop. $(Sp(v_1, \ldots, v_m), +, \cdot)$ is a subspace.

Pf. $0 \in Sp(v_1, \ldots, v_m)$ (by letting all $\alpha_i = 0$)

$$(\alpha_1 v_1 + \cdots + \alpha_m v_m) + (\alpha_1' v_1 + \cdots + \alpha_m' v_m)$$

$$= (\alpha_1 + \alpha_1') v_1 + \cdots + (\alpha_m + \alpha_m') v_m \in Sp(v_1, \ldots, v_m)$$

$$\gamma(\alpha_1 v_1 + \cdots + \alpha_m v_m) = \gamma\alpha_1 v_1 + \cdots + \gamma\alpha_m v_m \in Sp(v_1, \ldots, v_m)$$

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$\therefore$ $Sp(v_1, \ldots, v_m)$ is the smallest subspace that contains $v_1, \ldots, v_m$.

If $\mathcal{U}, \mathcal{V}$ are subspaces of $\mathbb{R}^n$,

ⓐ $\mathcal{U} \cap \mathcal{V}$ is subspace

ⓑ $\mathcal{U} + \mathcal{V} = \{ u + v \mid u \in \mathcal{U} \ \& \ v \in \mathcal{V} \}$ is a subspace

pf// ⓐ $p, q \in \mathcal{U} \cap \mathcal{V} \Rightarrow p+q \in \mathcal{U} \ \& \ p+q \in \mathcal{V} \Rightarrow p+q \in \mathcal{U} \cap \mathcal{V}$

$\alpha p \in \mathcal{U} \ \& \ \alpha p \in \mathcal{V} \Rightarrow \alpha p \in \mathcal{U} \cap \mathcal{V}$

& since $0 \in \mathcal{U}$, $0 \in \mathcal{V} \Rightarrow 0 \in \mathcal{U} \cap \mathcal{V}$ #

ⓑ $p, q \in \mathcal{U} + \mathcal{V}$

$$p = p_1 + p_2, \quad p_1 \in \mathcal{U}, \ p_2 \in \mathcal{V}$$

$$+ \quad q = q_1 + q_2, \quad q_1 \in \mathcal{U}, \ q_2 \in \mathcal{V}$$

$$p + q = \underset{\substack{\cap \\ \mathcal{U}}}{(p_1 + q_1)} + \underset{\substack{\cap \\ \mathcal{V}}}{(p_2 + q_2)} \in \mathcal{U} + \mathcal{V}$$

$$\alpha p = \underset{\substack{\cap \\ \mathcal{U}}}{\alpha p_1} + \underset{\substack{\cap \\ \mathcal{V}}}{\alpha p_2} \in \mathcal{U} + \mathcal{V}$$

$$0 = \underset{\substack{\cap \\ \mathcal{U}}}{0} + \underset{\substack{\cap \\ \mathcal{V}}}{0} \in \mathcal{U} + \mathcal{V}$$

#

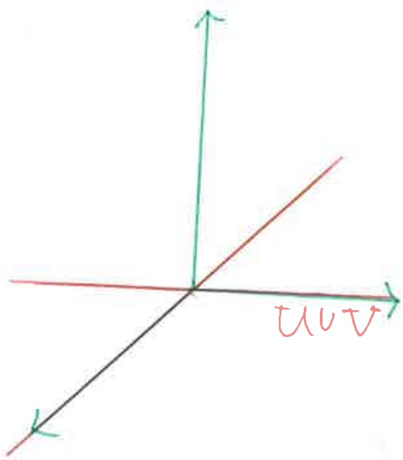
$\mathcal{U} \cup \mathcal{V}$ is not necessary a subspace.

in $\mathbb{R}^3$, $\mathcal{U} = Sp\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$ $\quad \mathcal{V} = Sp\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

$= x$ axis $\quad = y$ axis

$= \left\{ \begin{pmatrix}x\\0\\0\end{pmatrix} \right\}$ $\quad = \left\{ \begin{pmatrix}0\\y\\0\end{pmatrix} \right\}$

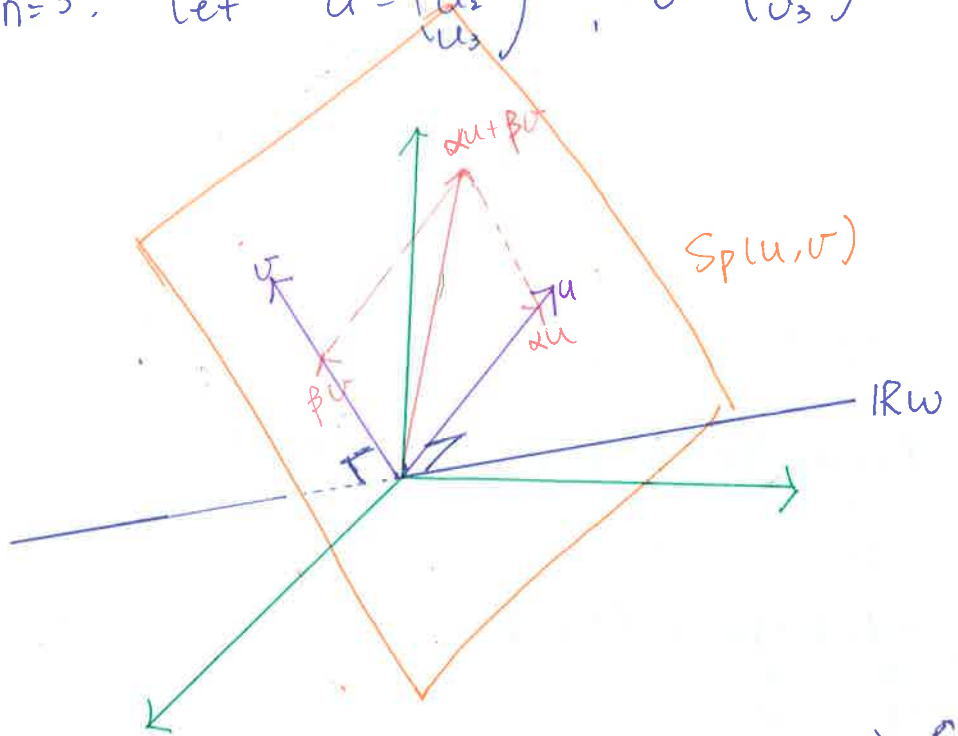
But $\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \notin \mathcal{U} \cup \mathcal{V}$



In $\mathbb{R}^n$, let $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \leftarrow i\text{TH}$ "standard basis vector"

$$\mathbb{R}^n = Sp(e_1, \dots, e_n)$$

$n=3$. let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $u \notin Sp(v)$ (ie $u \neq \lambda v$)



Find $w \in \mathbb{R}^3$ where $w \perp u$ & $w \perp v$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$w \perp u \Rightarrow w \cdot u = 0 \Rightarrow \begin{cases} w_1 u_1 + w_2 u_2 + w_3 u_3 = 0 \\ w_1 v_1 + w_2 v_2 + w_3 v_3 = 0 \end{cases}$$
$$w \perp v \Rightarrow w \cdot v = 0 \Rightarrow$$

since $u \neq \lambda v \Rightarrow$ one of w_1, w_2, w_3 is free, say $w_1 = 1$

$$\begin{cases} w_2 u_2 + w_3 u_3 = -u_1 \\ w_2 v_2 + w_3 v_3 = -v_1 \end{cases}$$

$$\Rightarrow \quad w_2 = \frac{-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}} \ ; \quad w_3 = \frac{\begin{vmatrix} u_2 & -u_1 \\ v_2 & -v_1 \end{vmatrix}}{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}} = \frac{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}$$

$$\therefore \quad w = \frac{1}{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \\ -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \\ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{pmatrix} \rightarrow u \times v$$

cross product
exterior product
wedge product

Wedge product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$(u, v) \mapsto u \times v$

if $\mathbb{R}^n = Sp\{p_1, \dots, p_n\}$

$u, v \in Span\{p_1, \dots, p_{n-1}\}$

$u \times v$ gives a vector $\perp$ to all vectors in $Sp\{p_1, \dots, p_{n-1}\}$.

(exercise: write down the formula)

Easy to check

Easy to check:

- $(\alpha u) \times w = \alpha(u \times w) = u \times (\alpha w)$
- $(u+v) \times w = u \times w + v \times w$
- $u \times v = -v \times u$.

$(\Rightarrow \lambda u \times u = -u \times \lambda u = -\lambda u \times u \Rightarrow \lambda u \times u = 0)$

cross product measures how far two vectors are from parallel.

HW: $n=3$ $\|u \times v\| = \|u\| \, \|v\| \sin\theta$

In $\mathbb{R}^3$, the direction of 11/7 Ends ...rmined by right hand rule

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$\therefore$ a hyperplane in $\mathbb{R}^3$ spanned by u, v is described by

$H = \{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{R}\}$ $\longleftrightarrow$ parametric eq. (describe each coordinate individually)

OR

normal vector

$H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid ax + by + cz = 0, \ \begin{pmatrix} a \\ b \\ c \end{pmatrix} = u \times v \right\}$ $\leftarrow$ trace of an equation

Shifting by $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ we get eq. of affine plane

spanning vectors

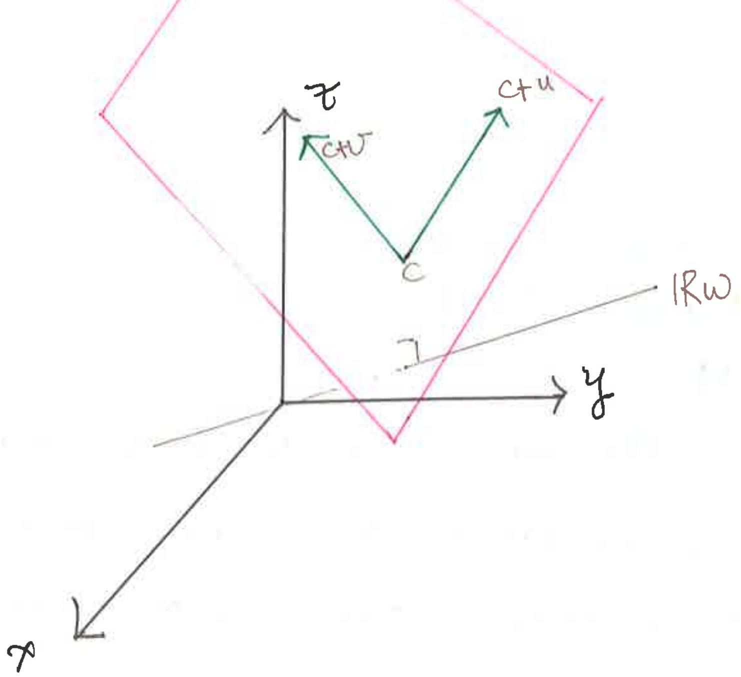
$$H_c = \{ c + \alpha u + \beta v \mid \alpha, \beta \in \mathbb{R} \} \quad \text{OR}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; a(x-c_1) + b(y-c_2) + c(z-c_3) = 0 \right\}$$

OR $ax + by + cz + d = 0$, $d = -ac_1 - bc_2 - cc_3$

where $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = u \times v$

Note $u, v \notin H_c$ if $c \notin H$, But $u \times v$ still gives a normal vector to H_c



eg. Find parametric / xyz equations for H_c with $c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, & spanning vectors $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Parametric: $H_c = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$

$$= \left\{ \begin{pmatrix} 1+\alpha+\beta \\ 1+\alpha+\beta \\ 1+\alpha \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

xyz: $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$H_c = \{ (x, y, z) \mid x + y = 0 \}$

$\Rightarrow -x + y + d = 0$, $c \in H_c \Rightarrow -1 + 1 + d = 0 \Rightarrow d =$

eg11 Find parametric eq. of the plane w/ xyz equation $x+y+z+1=0$

$w = \begin{pmatrix}1\\1\\1\end{pmatrix}$, need two vectors u, v so that u, v, w are mutually $\perp$

take $u = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$v = w \times u = \left(\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix}, -\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix}, \begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} \right)^T$$

$$= \begin{pmatrix}-1\\2\\-1\end{pmatrix}$$; and a point on H_c is $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\therefore \quad H = \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha \begin{pmatrix}1\\0\\-1\end{pmatrix} + \beta \begin{pmatrix}-1\\2\\-1\end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$