**Thm.** $(\mathbb{R}, |\cdot|)$ is complete

$\Updownarrow$

Every bounded monotonic sequence converges
(Monotone convergent theorem)

**pf.** $\Downarrow$ done in class.

$\Uparrow$ Let $\{a_n\}$ be Cauchy, and so bounded.

$\therefore \limsup_n a_n = \lim_{k\to\infty} \underbrace{\sup_{n\geq k}\{a_n\}}_{=b_k} = b \in \mathbb{R}$, $\quad b_k \geq b_{k+1} \ \forall k$

$\hspace{27em} \rotatebox{90}{$\leq$}$

and $\liminf_n a_n = \lim_{k\to\infty} \underbrace{\inf_{n\geq k}\{a_n\}}_{c_k} = c \in \mathbb{R}$, $\quad c_k \leq c_{k+1} \ \forall k$

It suffices to show $\underline{\underline{b=c}}$.

(Recall: $a_n \to L \iff \liminf a_n = \limsup a_n = L$ )

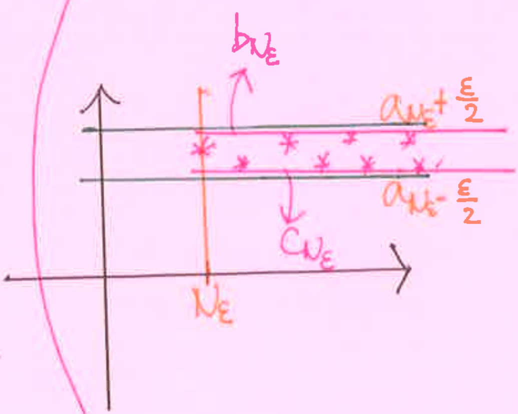
$\{a_n\}$ Cauchy, $\therefore$ $\forall\, \varepsilon>0, \ \exists\, N_\varepsilon \in \mathbb{N}$ s.t.

$$|a_n - a_m| < \frac{\varepsilon}{2} \qquad \forall\, n, m \geq N_\varepsilon.$$

In particular,

$$a_{N_\varepsilon} - \frac{\varepsilon}{2} < a_n < a_{N_\varepsilon} + \frac{\varepsilon}{2} \qquad \forall\, n \geq N_\varepsilon$$

$\therefore$ $a_{N_\varepsilon} + \frac{\varepsilon}{2}$ is an upper bound for $\{a_n\}_{n\geq N_\varepsilon}$




$$\Rightarrow \quad b_{N_\varepsilon} = \sup_{n\geq N_\varepsilon}\{a_n\} \leq a_{N_\varepsilon} + \frac{\varepsilon}{2}$$

Similarly, $c_{N_\varepsilon} \geq a_{N_\varepsilon} - \frac{\varepsilon}{2}$

$$\Rightarrow \quad 0 \leq b_{N_\varepsilon} - c_{N_\varepsilon} \leq a_{N_\varepsilon} + \frac{\varepsilon}{2} - \left(a_{N_\varepsilon} - \frac{\varepsilon}{2}\right) = \varepsilon$$

since $b_k$ decreasing & $c_k$ increasing, we have $\forall k >$

$$\boxed{0 \leq b_k - c_k \leq b_{N_\varepsilon} - c_{N_\varepsilon} \leq \varepsilon \quad \forall\, k > N_\varepsilon}$$

$b_k - c_k \to 0$

$\Rightarrow b = c$

QED.