HW 11.

1. Let $f(x) = \begin{cases} e^{\frac{-1}{x}}, & x>0, \\ 0, & x \le 0. \end{cases}$

Prove that $f^{(n)}(x) = \begin{cases} \dfrac{P_{n-1}(x)}{x^{2n}} \cdot e^{\frac{-1}{x}}, & x>0, \\ 0, & x \le 0, \end{cases}$

where $P_{n-1}(x)$ is a polynomial with degree $n-1$.

Prove that $f \in C^{\infty}(\mathbb{R})$.

<pf>

① As $x>0$, then $\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^{\frac{-1}{x}}) = e^{\frac{-1}{x}} \cdot \frac{d}{dx}\left(\frac{-1}{x}\right) = \frac{1}{x^2} e^{\frac{-1}{x}}$.

As $x<0$, then $\frac{d}{dx}(f(x)) = \frac{d}{dx}(0) = 0$.

As $x=0$, then

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

(1)

$$\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{e^{\frac{-1}{h}}}{h} = \lim_{h\to 0^+} \frac{\frac{1}{h}}{e^{\frac{1}{h}}} \overset{L'H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{h\to 0^+} \frac{\frac{-1}{h^2}}{\frac{-1}{h^2} e^{\frac{1}{h}}}$$

$$= \lim_{h\to 0^+} \frac{1}{e^{\frac{1}{h}}} = 0. \quad \left(\text{as } h\to 0^+,\ \frac{1}{h}\to +\infty,\ e^{\frac{1}{h}} \to +\infty\right)$$

(2)

$$\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0. \quad \text{So } f'(0) = 0.$$

Thus $f^{(1)}(x) = \begin{cases} \frac{1}{x^2} e^{\frac{-1}{x}} & , x>0 \\ 0 & , x \le 0 . \end{cases}$

Let $P_0(x) = 1$. Then $P_0(x)$ is a polynomial and $\deg P_0 = 0$.

$$\text{So } f^{(1)}(x) = \begin{cases} \frac{P_0(x)}{x^2} e^{\frac{-1}{x}} & , x>0 \\ 0 & , x \le 0 . \end{cases}$$

Suppose $f^{(n)}(x) = \begin{cases} \frac{P_{n-1}(x)}{x^{2n}} e^{\frac{-1}{x}} , & x>0 \\ 0 , & x \le 0 \end{cases}$, where $P_{n-1}(x)$ is a polynomial with $\deg P_{n-1} = n-1$.

As $x>0$, then

$$f^{(n+1)}(x) = \frac{d}{dx}\left(f^{(n)}(x)\right) = \frac{d}{dx}\left(\frac{P_{n-1}(x)}{x^{2n}} e^{\frac{-1}{x}}\right)$$

$$= \frac{P_{n-1}'(x) \cdot x^{2n} - P_{n-1}(x) \cdot 2n \cdot x^{2n-1}}{x^{4n}} \cdot e^{\frac{-1}{x}} + \frac{P_{n-1}(x)}{x^{2n}} \cdot \frac{1}{x^2} e^{\frac{-1}{x}}$$

$$= \left(\frac{P_{n-1}'(x)}{x^{2n}} - \frac{2n P_{n-1}(x)}{x^{2n+1}} + \frac{P_{n-1}(x)}{x^{2n+2}} \right) e^{\frac{-1}{x}}$$

$$= \frac{x^2 \cdot P_{n-1}'(x) - 2nx P_{n-1}(x) + P_{n-1}(x)}{x^{2n+2}} \cdot e^{\frac{-1}{x}}$$

Let $P_n(x) = x^2 \cdot P_{n-1}'(x) - 2nx P_{n-1}(x) + P_{n-1}(x)$. Then $P_n(x)$ is a polynomial with $\deg P_n = n$.

So $f^{(n+1)}(x) = \frac{P_n(x)}{x^{2n+2}} \cdot e^{\frac{-1}{x}}$, where $P_n(x)$ is a polynomial and $\deg P_n = n$.

As $x<0$, then

$$f^{(n+1)}(x) = \frac{d}{dx}\left(f^{(n)}(x)\right) = \frac{d}{dx}(0) = 0.$$

As $x=0$, then $f^{(n+1)}(0) = \lim\limits_{h\to 0} \dfrac{f^{(n)}(h) - f^{(n)}(0)}{h} = \lim\limits_{h\to 0} \dfrac{f^{(n)}(h)}{h}$.

(1)

$$\lim_{h\to 0^+} \frac{f^{(n)}(h)}{h} = \lim_{h\to 0^+} \frac{\frac{P_{n-1}(h)}{h^{2n}} e^{\frac{-1}{h}}}{h} = \lim_{h\to 0^+} \frac{P_{n-1}(h)}{h^{2n+1}\cdot e^{\frac{1}{h}}} = \lim_{h\to 0^+} \frac{\overbrace{\frac{P_{n-1}(h)}{h^{2n+1}}}^{\text{degree}=n-1}}{e^{\frac{1}{h}}}$$

$$\underset{(\frac{\infty}{\infty})}{\overset{L'H'}{=}} \lim_{h\to 0^+} \frac{\frac{P'_{n-1}(h)\,h^{2n+1} - (2n+1)h^{2n}\cdot P_{n-1}(h)}{h^{4n+2}}}{\frac{-1}{h^2} e^{\frac{1}{h}}} = \lim_{h\to 0^+} \frac{\frac{(2n+1)\cdot h^{2n} P_{n-1}(h)}{h^{4n}} - \frac{h^{2n+1}\cdot P'_{n-1}(h)}{h^{4n}}}{e^{\frac{1}{h}}}$$

$$= \lim_{h\to 0^+} \frac{\frac{\overbrace{\left[(2n+1)P_{n-1}(h) - h\cdot P'_{n-1}(h)\right]}^{\text{degree}=n-1}}{h^{2n}}}{e^{\frac{1}{h}}}$$

$$\underset{(\frac{\infty}{\infty})}{\overset{L'H'}{=}} \lim_{h\to 0^+} \frac{\frac{h^{2n}\left[(2n+1)P'_{n-1}(h) - h\,P''_{n-1}(h)\right] - \left[(2n+1)P_{n-1}(h) - h\,P'_{n-1}(h)\right]\cdot 2n\,h^{2n-1}}{h^{4n}}}{\frac{-1}{h^2} e^{\frac{1}{h}}}$$

$$= \lim_{h\to 0^+} \frac{\frac{\overbrace{\left[(2n)(2n+1)P_{n-1}(h) - 2nh\,P'_{n-1}(h) - (2n+1)h\,P'_{n-1}(h) + h\,P''_{n-1}(h)\right]}^{\text{degree}=n-1}}{h^{2n-1}}}{e^{\frac{1}{h}}}$$

會發現「分子的分數」的分母次方越來越小，所以 L'H' 做了 $2n+1$ 次後，會有如下：

$$= \lim_{h\to 0^+} \frac{[\text{degree}=n-1\text{的多項式}]}{e^{\frac{1}{h}}} = 0 \quad \left(\because \text{ as } h\to 0^+,\ \frac{1}{h}\to +\infty \Rightarrow e^{\frac{1}{h}} \to +\infty\right)$$

(2)

$$\lim_{h\to 0^-} \frac{f^{(n)}(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0. \text{ Thus } f^{(n+1)}(0) = \lim_{h\to 0} \frac{f^{(n)}(h)}{h} = 0.$$

Hence
$$f^{(n+1)}(x) = \begin{cases} \dfrac{P_n(x)}{x^{2n+2}}\cdot e^{\frac{-1}{x}}, & x>0, \text{ where } P_n(x) \text{ is a polynomial with } \deg P_n = n. \\ 0, & x\le 0 \end{cases}$$

By induction, we complete this proof. $\blacksquare$

③

$$f^{(n)}(x) = \begin{cases} \dfrac{P_{n-1}(x)}{x^{2n}} e^{\frac{-1}{x}}, & x>0, \text{ where } P_{n-1}(x) \text{ is a polynomial with } \deg P_{n-1} = n-1. \\ 0, & x \le 0 \end{cases}$$

For each $n \in \mathbb{N}$.

Since x^{2n} and $P_{n-1}(x)$ are continuous, then $\dfrac{P_{n-1}(x)}{x^{2n}}$ is also continuous as $x>0$.

Since constant function is continuous as $x<0$, then $f^{(n)}(x)$ is continuous on $\mathbb{R}$.

Thus $f \in C^{\infty}(\mathbb{R})$. $\blacksquare$

2. Salas §12-9 ※2, 8, 14, 15, 24, 44, 47, 49, 51

2. $f(x) = \dfrac{1}{(1-x)^3}$

<sol>

Since $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$ and $\dfrac{d^2}{dx^2}\left(\dfrac{1}{1-x}\right) = \dfrac{d}{dx}\left(\dfrac{1}{(1-x)^2}\right) = \dfrac{2}{(1-x)^3}$,

then $\dfrac{1}{(1-x)^3} = \dfrac{1}{2}\dfrac{d^2}{dx^2}\left(\dfrac{1}{1-x}\right)$

$= \dfrac{1}{2}\dfrac{d^2}{dx^2}\left[1+x+x^2+x^3+\cdots+x^n+\cdots\right], \quad -1<x<1$

$= \dfrac{1}{2}\dfrac{d}{dx}\left[1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots\right], \quad -1<x<1$

$= \dfrac{1}{2}\left(2+6x+12x^2+20x^3+\cdots+n(n-1)x^{n-2}+\cdots\right), \quad -1<x<1$

※

8. $f(x) = \ln(\cos x)$

<sol>

Since $\dfrac{d}{dx}(\ln(\cos x)) = \dfrac{-\sin x}{\cos x} = -\tan x$, then $\ln(\cos x) = \int -\tan x\, dx$.

$\ln(\cos x) = -\int \tan x\, dx = -\int\left(x+\dfrac{1}{3}x^3+\dfrac{2}{15}x^5+\dfrac{17}{315}x^7+\cdots\right)dx$

$= -\dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 - \dfrac{2}{90}x^6 - \dfrac{17}{2520}x^8 - \cdots + C$,

C = constant.

$x=0 \Rightarrow \ln(\cos 0) = C$

$\Rightarrow C = \ln 1 = 0.$

$\therefore \ln(\cos x) = -\dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 - \dfrac{1}{45}x^6 - \dfrac{17}{2520}x^8 - \cdots$

※

14、 $f(x)=\frac{1-x}{1+x}$

<sol>

$$f(x)=\frac{1-x}{1+x}=\frac{1}{1+x}-x\cdot\frac{1}{1+x}\quad,\ \text{as } x\neq -1$$

$$=\frac{1}{1-(-x)}-x\cdot\frac{1}{1-(-x)}\quad,\ -1<x<1$$

$$=\sum_{k=0}^{\infty}(-x)^k-x\cdot\sum_{k=0}^{\infty}(-x)^k\ ,\ -1<x<1$$

$$=\sum_{k=0}^{\infty}(-1)^k\cdot x^k-\sum_{k=0}^{\infty}(-1)^k\cdot x^{k+1}\ ,\ -1<x<1$$

$$=1+\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}+\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}\ ,\ -1<x<1.$$

$$=1+2\cdot\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}\quad,\ -1<x<1$$

15 $f(x)=\frac{2x}{1-x^2}$

<sol>

$$f(x)=2x\cdot\frac{1}{1-x^2}=2x\cdot\sum_{k=0}^{\infty}(x^2)^k\quad,\ 0\leq x^2<1 \iff -1<x<1$$

$$=\sum_{k=0}^{\infty}2x^{2k+1}\ ,\ -1<x<1$$

44. $f(x) = \dfrac{e^x - 1}{x}$, $x \neq 0$, $x \in \mathbb{R}$

(a)

Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ for each $x \in \mathbb{R}$,

then $\dfrac{e^x - 1}{x} = 1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots$ for each $x \in \mathbb{R}$, $x \neq 0$.

※

(b) <pf>

By The Differentiability Thm (12-9-2), $\dfrac{e^x - 1}{x} = 1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots$ for $x \in \mathbb{R}$, $x \neq 0$,

then $\dfrac{d}{dx}\left(\dfrac{e^x - 1}{x}\right) = \dfrac{d}{dx}\left(1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots + \dfrac{x^n}{(n+1)!} + \cdots\right)$, $x \in \mathbb{R} \setminus \{0\}$.

$\Rightarrow \dfrac{x \cdot e^x - (e^x - 1)}{x^2} = \dfrac{1}{2!} + \dfrac{2}{3!}x + \dfrac{3}{4!}x^2 + \cdots + \dfrac{n}{(n+1)!}x^{n-1} + \cdots$

$\Rightarrow \dfrac{x \cdot e^x - e^x + 1}{x^2} = \dfrac{1}{2!} + \dfrac{2}{3!}x + \dfrac{3}{4!}x^2 + \cdots + \dfrac{n}{(n+1)!}x^{n-1} + \cdots$

$x = 1$, then $\dfrac{1 \cdot e^1 - e^1 + 1}{1^2} = \dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n+1)!} + \cdots$

$\Rightarrow 1 = \displaystyle\sum_{n=1}^{\infty} \dfrac{n}{(n+1)!}$

24.

$$\lim_{x\to 0}\frac{\sin x - x}{x^2}$$

<sol>

(i)

$$\lim_{x\to 0}\frac{\sin x - x}{x^2} \underset{(\frac{0}{0})}{\overset{L'H.}{=}} \lim_{x\to 0}\frac{\cos x - 1}{2x} \underset{(\frac{0}{0})}{\overset{L'H.}{=}} \lim_{x\to 0}\frac{-\sin x}{2} = 0$$

(ii)

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\cdots$ for all $x \in \mathbb{R}$,

then $\frac{\sin x - x}{x^2} = -\frac{x}{3!} + \frac{x^3}{5!} - \frac{x^5}{7!} + \frac{x^7}{9!} - \cdots\cdots$ for each $x \in \mathbb{R}$, $x \neq 0$.

So $\lim_{x\to 0}\frac{\sin x - x}{x^2} = \lim_{x\to 0}\left(\frac{-x}{3!} + \frac{x^3}{5!} - \frac{x^5}{7!} + \frac{x^7}{9!} - \cdots\cdots\right)$ (is polynomial)

$$= 0$$

47. If $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ both converge to the same sum on some interval, then $a_k = b_k$ for each k.

<pf> Let $f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be defined on $I = (-c, c)$, $c \underset{\neq 0}{\in} \mathbb{R}$.

By the differentiablity theorem (12-9-2), then we have

$$f^{(1)}(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \text{ converges on } I.$$

$\Rightarrow f(x)$ is differentiable on I and $a_1 = f^{(1)}(0)$.

By the differentiablity theorem (12-9-2), then we have

$$f^{(2)}(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(k a_k x^{k-1}) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots \text{ converges on } I.$$

$\Rightarrow f^{(1)}(x)$ is differentiable on I and $2a_2 = f^{(2)}(0)$. $\Rightarrow a_2 = \frac{f^{(2)}(0)}{2!}$

By the differentiablity theorem (12-9-2), then we have

$$f^{(3)}(x) = 6a_3 + 24a_4 x + 60 a_5 x^2 + \cdots \text{ converges on } I.$$

$\Rightarrow f^{(2)}(x)$ is differentiable on I and $6a_3 = f^{(3)}(0)$. $\Rightarrow a_3 = \frac{f^{(3)}(0)}{6} = \frac{f^{(3)}(0)}{3!}$

Step by Step, we have $a_k = \frac{f^{(k)}(0)}{k!}$ for each k. and f is infinitely differentiable on I.

Since $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k$, then we have $f(x) = \sum_{k=0}^{\infty} b_k x^k$ for each $x \in I = (-c, c)$, $c \underset{\neq 0}{\in} \mathbb{R}$.

By above discussion, we have $b_k = \frac{f^{(k)}(0)}{k!}$ for each k.

So $a_k = b_k$ for each k. ■

49. $f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots$

(a) Show that if f is an even function, then $a_{2k+1} = 0$ for all k.

(b) Show that if f is an odd function, then $a_{2k} = 0$ for all k.

<pf> Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be defined on I, $I = (-c, c)$, $c \neq 0 \in \mathbb{R}$.

By problem 47, then we have $f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, $x \in I$.

① Suppose f is differentiable and f is even. Claim: f' is odd.

Since $f(x) = f(-x)$, then $f'(x) = -f'(-x)$. $\Rightarrow$ f' is odd.

② Suppose f is differentiable and f is odd. Claim: f' is even.

Since $f(x) = -f(-x)$, then $f'(x) = (-f'(-x)) \cdot (-1) = f'(-x)$. $\Rightarrow$ f' is even.

(a)

If f is even, by ①, then $f^{(1)}(x)$, $f^{(3)}(x)$, $f^{(5)}(x)$, $\cdots\cdots$, $f^{(2k-1)}(x)$, $\cdots$ are odd, $k \in \mathbb{N}$.

$\Rightarrow f^{(2k-1)}(0) = 0$ for each $k \in \mathbb{N}$.

$\Rightarrow a_{2k-1} = \frac{f^{(2k-1)}(0)}{(2k-1)!} = 0$ for each $k \in \mathbb{N}$.

(b)

If f is odd, by ②, then $f^{(2)}(x)$, $f^{(4)}(x)$, $f^{(6)}(x)$, $\cdots$, $f^{(2k)}(x)$, $\cdots\cdots$ are odd, $k \in \mathbb{N}$.

$\Rightarrow f^{(2k)}(0) = 0$ for each $k \in \mathbb{N}$.

$\Rightarrow a_{2k} = \frac{f^{(2k)}(0)}{(2k)!} = 0$ for each $k \in \mathbb{N}$.

■

51. Suppose that f is infinitely differentiable on an open interval that contains 0, and suppose $f''(x) = -2f(x)$, $f(0) = 0$, $f'(0) = 1$.

Express $f(x)$ as a power series in x. What is the sum of this series?

<sol>

$$f''(x) = -2f(x) \Rightarrow f''(0) = -2f(0) = 0$$

$$f'''(x) = -2f'(x) \Rightarrow f'''(0) = -2f'(0) = -2$$

$$f^{(4)}(x) = -2f^{(2)}(x) \Rightarrow f^{(4)}(0) = -2f^{(2)}(0) = 0$$

$$f^{(5)}(x) = -2f^{(3)}(x) \Rightarrow f^{(5)}(0) = -2f^{(3)}(0) = -2\times -2 = 4$$

$$f^{(6)}(x) = -2f^{(4)}(x) \Rightarrow f^{(6)}(0) = -2f^{(4)}(0) = 0$$

$$f^{(7)}(x) = -2f^{(5)}(x) \Rightarrow f^{(7)}(0) = -2f^{(5)}(0) = -2\times 4 = -8$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Since f is infinitely differentiable on an open interval I and $0 \in I$,

by Taylor's Theorem, then

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^{(7)}(0)}{7!}x^7 + \cdots\cdots, \quad x \in I.$$

$$\Rightarrow f(x) = x - \frac{2}{3!}x^3 + \frac{4}{5!}x^5 - \frac{8}{7!}x^7 + \cdots\cdots = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^k}{(2k+1)!} x^{2k+1}, \quad x \in I.$$

Since $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$ for each $x \in I$,

then $x - \frac{2}{3!}x^3 + \frac{4}{5!}x^5 - \frac{8}{7!}x^7 + \cdots = \frac{1}{\sqrt{2}}\left(\sqrt{2}x - \frac{1}{3!}(\sqrt{2}x)^3 + \frac{1}{5!}(\sqrt{2}x)^5 - \frac{1}{7!}(\sqrt{2}x)^7 + \cdots\right)$

$$= \frac{1}{\sqrt{2}}\sin(\sqrt{2}x) \text{ for each } x \in I.$$