Hw12

1. Salas §3-7 # 20, 53, 58, 10.

10. $\tan(xy) = xy$

<sol>

$$\frac{d\tan(xy)}{dx} = \frac{d(xy)}{dx}$$

$$\sec^2(xy)\cdot\frac{d(xy)}{dx} = \frac{d(xy)}{dx} \Rightarrow \tan^2(xy)\cdot\frac{d(xy)}{dx} = 0$$

$$\Rightarrow (xy)^2\cdot\left[y + x\frac{dy}{dx}\right] = 0$$

$$\Rightarrow y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

#

20. $x = \sin^2 y$, $\left(\frac{1}{2}, \frac{\pi}{4}\right)$

<sol>

$$\frac{dx}{dx} = \frac{d(\sin^2 y)}{dx}$$

$$1 = 2\sin y\cos y\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sin 2y} \qquad \left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},\frac{\pi}{4}\right)} = \frac{1}{\sin\frac{\pi}{2}} = 1$$

#

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sin 2y}\right) = \frac{-2\cos 2y}{\sin^2 2y}\cdot\frac{dy}{dx} = \frac{-2\cos 2y}{\sin^2 2y\cdot\sin 2y} = \frac{-2\cos 2y}{\sin^3 2y}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{1}{2},\frac{\pi}{4}\right)} = \frac{0}{1^3} = 0$$

#

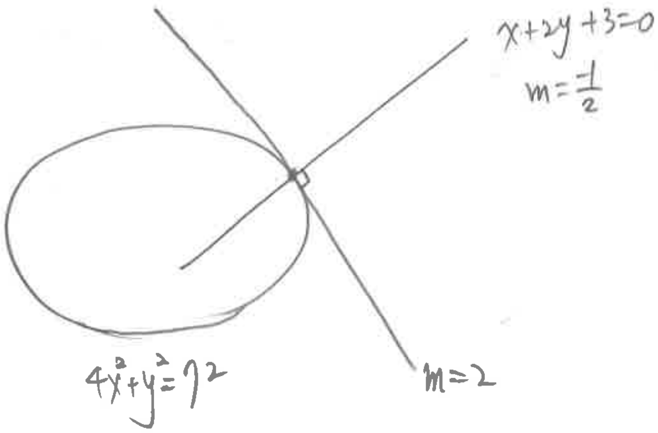
53.

$$4x^2+y^2=72$$

<sol>

$$\frac{d(4x^2+y^2)}{dx}=\frac{d(72)}{dx}$$

$$8x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-4x}{y}=2 \Rightarrow y=-2x$$



$$\begin{cases} 4x^2+y^2=72 \\ y=-2x \end{cases} \Rightarrow 4x^2+4x^2=72 \Rightarrow x^2=9 \Rightarrow x=\pm 3$$

$$\therefore \begin{cases} x=3 \\ y=-6 \end{cases} \text{ and } \begin{cases} x=-3 \\ y=6 \end{cases}$$

$\therefore$ tangent line at $(3,-6)$ is $y=2(x-3)-6$

tangent line at $(-3,6)$ is $y=2(x+3)+6$ $\#$

58.

$$\begin{cases} x^2+(y-a)^2=1, & a>0 \\ y=2x^2 \end{cases}$$

<sol>

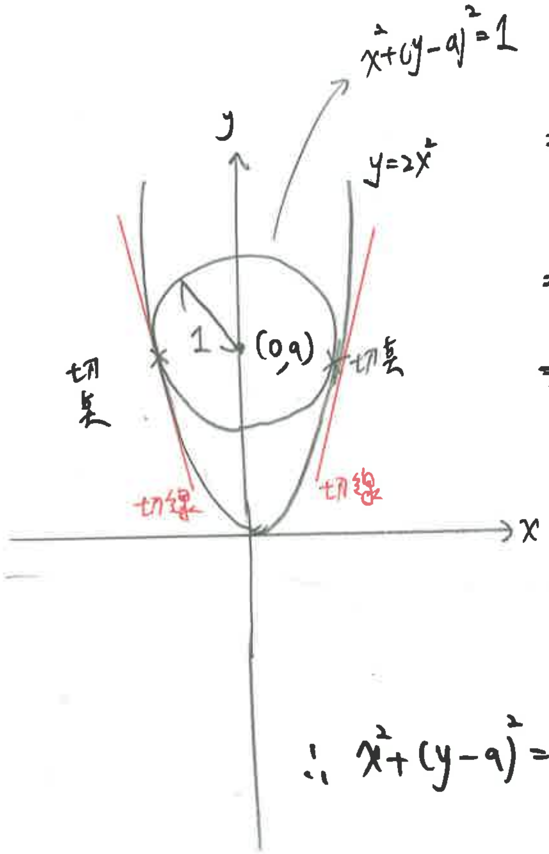
$x^2+(y-a)^2=1$

$\Rightarrow 2x+2(y-a)\dfrac{dy}{dx}=0$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{-x}{y-a}$

$y=2x^2$

$\Rightarrow \dfrac{dy}{dx}=4x$



$\Rightarrow \dfrac{-x}{y-a}=4x$

$\Rightarrow 4x(y-a)=-x$

$\Rightarrow 4y-4a=-1$, if $x\neq 0$

$\Rightarrow y=\dfrac{4a-1}{4}$

$\Rightarrow x^2=\dfrac{4a-1}{8}$

$\therefore x^2+(y-a)^2=1 \Rightarrow \dfrac{4a-1}{8}+\left(\dfrac{-1}{4}\right)^2=1 \Rightarrow \dfrac{4a-1}{8}=\dfrac{15}{16}$

$\Rightarrow 8a-2=15 \Rightarrow a=\dfrac{17}{8} \Rightarrow \begin{cases} y=\dfrac{15}{8} \\ x=\pm\dfrac{\sqrt{15}}{4} \end{cases}$

$\therefore$ tangent point are $\left(\dfrac{\sqrt{15}}{4},\dfrac{15}{8}\right)$ or $\left(\dfrac{-\sqrt{15}}{4},\dfrac{15}{8}\right)$

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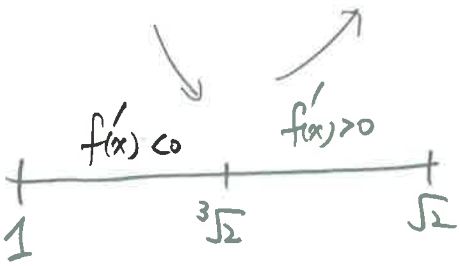
2, Salas §4-4 ※8, 18, 41

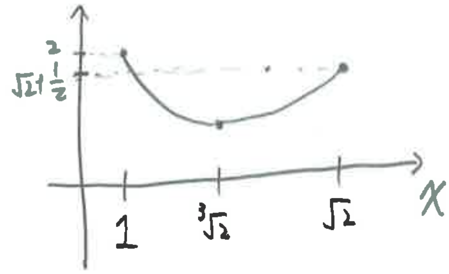
8、

$$f(x) = x + \frac{1}{x^2}, \quad x \in [1, \sqrt{2}]$$

<sol>



$$f'(x) = 1 + \frac{-2x}{x^4} = 1 - \frac{2}{x^3}$$



Let $f'(x) = 0 \Rightarrow \frac{2}{x^3} = 1 \Rightarrow x^3 = 2 \Rightarrow \underline{x = \sqrt[3]{2}}$ (critical point)

$f(1) = 2$ (endpoint maximum)

$f(\sqrt[3]{2}) = \sqrt[3]{2} + 2^{\frac{-2}{3}} = \sqrt[3]{2} + \frac{1}{\sqrt[3]{4}} = \frac{3}{\sqrt[3]{4}} = 3 \times (2)^{\frac{-2}{3}}$ (local minimum)

$f(\sqrt{2}) = \sqrt{2} + \frac{1}{2}$ (endpoint maximum)

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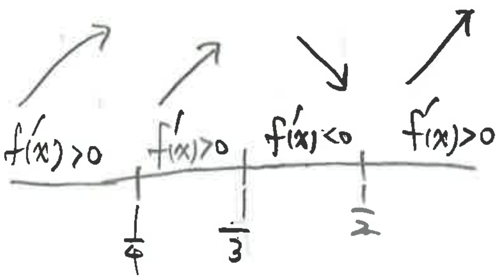
18、 $f(x) = (4x-1)^{\frac{1}{3}} \cdot (2x-1)^{\frac{2}{3}}$

<sol>

$$f'(x) = \frac{4}{3}(4x-1)^{\frac{-2}{3}} \cdot (2x-1)^{\frac{2}{3}} + \frac{4}{3}(4x-1)^{\frac{1}{3}} \cdot (2x-1)^{\frac{-1}{3}}$$

$$= \frac{4}{3}(4x-1)^{\frac{-2}{3}}(2x-1)^{\frac{-1}{3}}\left[(2x-1) + (4x-1)\right]$$

$$= \frac{4}{3} \cdot \frac{6x-2}{(4x-1)^{\frac{2}{3}} \cdot (2x-1)^{\frac{1}{3}}}$$

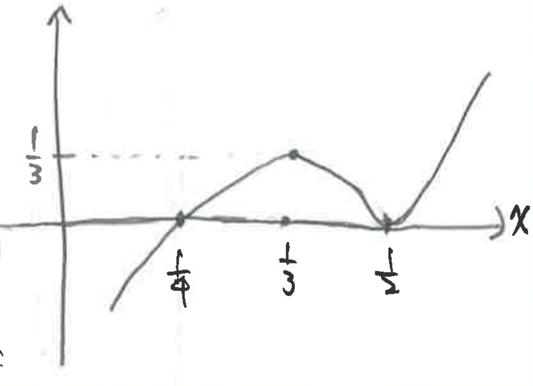


Let $f'(x) = 0 \Rightarrow \underline{x = \frac{1}{3}, \text{ but } x \neq \frac{1}{4}, x \neq \frac{1}{2}}$ (critical points)

$f(\frac{1}{4}) = 0$ (no extreme value)

$f(\frac{1}{3}) = \frac{1}{3}$ (local maximum)

$f(\frac{1}{2}) = 0$ (local minimum)



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41.

Given an example of a non constant function that takes on both its absolute maximum and absolute minimum on every interval.

Ex:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

(Dirichlet function)

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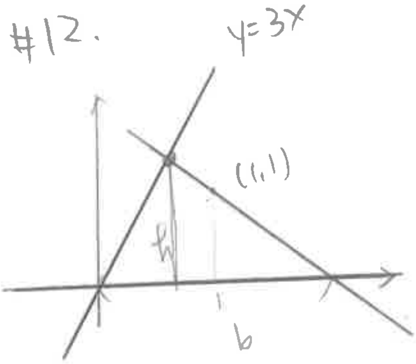
3. Salas §4-5 # 12, 18, 25, 35, 47, 55

HW12

3. §4.5

#12.



The equation of the third side is :

$y-1 = m(x-1) \Rightarrow y = mx + (1-m)$

The base of the triangle is :

$b = \dfrac{m-1}{m}$

The two lines intersect when $3x = mx + (1-m)$

$\Rightarrow x = \dfrac{1-m}{3-m}$

hence $h = \dfrac{3(1-m)}{3-m}$

Now we have the area of triangle :

$A(m) = \frac{1}{2}bh = \frac{1}{2}\left(\frac{m-1}{m}\right)\left(\frac{3(1-m)}{3-m}\right) = \dfrac{3(m+1)^2}{2m(m-3)}$ $m<0$

$\Rightarrow A'(m) = \frac{3}{2}\left(\dfrac{2(m+1)m(m-3) - (m-1)^2(2m-3)}{m^2(m-3)^2}\right) = \frac{-3}{2}\left(\dfrac{(m+3)(m-1)}{m^2(m-3)^2}\right)$

If $A'(m)=0 \Rightarrow m=-3$

hence the area of the triangle is minimum when the slope is -3.

#18 Find the points of the parabola $y=\frac{1}{8}x^2$ closest to $(0,6)$

It's sufficient to minimize the square of the distance :

$S = (x-0)^2 + (y-6)^2 = x^2 + (y-6)^2$

And x, y satisfies the equation $y = \frac{1}{8}x^2$ where $y \geq 0$.

$\Rightarrow x^2 = 8y$

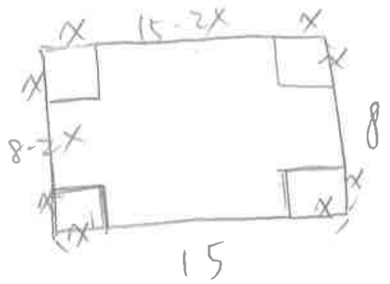
hence $S = x^2 + (y-6)^2 = 8y + (y-6)^2$

$\Rightarrow S'(y) = 8 + 2(y-6) = 2y-4$

If $S'(y) = 0 \rightarrow y = 2 \Rightarrow x = \pm\sqrt{16} = 4, -4$

Hence the points on the parabola that are closest to $(0,6)$ are $(4,2)$ and $(-4,2)$

#25



Let $V(x) = x(8-2x)(15-2x)$

We have $\begin{cases} x \ge 0 \\ 8-2x \ge 0 & \Rightarrow x \le 4 \\ 15-2x \ge 0 & \Rightarrow x \le \frac{15}{2} \end{cases} \Rightarrow 0 \le x \le 4.$

We want to maximize V.

$\Rightarrow V(x) = 120x - 46x^2 + 4x^3 \qquad 0 \le x \le 4$

$V'(x) = 120 - 92x + 12x^2$

$= 4(3x^2 - 23x + 30)$

$= 4(3x-5)(x-6)$

If $V'(x) = 0 \Rightarrow x = \frac{5}{3}$ or 6

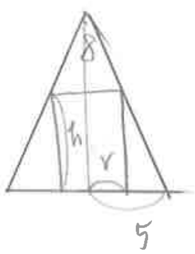
V': $+$ on $(0, \frac{5}{3})$, $-$ on $(\frac{5}{3}, 4)$; marks at $0, \frac{5}{3}, 4, 6$

hence $V(x)$ increases on $(0, \frac{5}{3})$ and decreases on $[\frac{5}{3}, 4)$

$\Rightarrow V(x)$ has maximum when $x = \frac{5}{3}$

Therefore the box of maximal volume is made by cutting out squares $\frac{5}{3}$ inches on a side.

#35.



We want to maximize V.

$\because V = \pi r^2 h$

By similar triangle, $\frac{8-h}{8} = \frac{r}{5} \Rightarrow 40 - 5h = 8r$

$\Rightarrow h = \frac{40-8r}{5}$

hence $V(r) = \pi r^2 h = \left(\frac{40r^2 - 8r^3}{5}\right)\pi$, $0 \le r \le 5$.

$\Rightarrow V'(r) = \left(\frac{80r - 24r^2}{5}\right)\pi$

If $V'(r) = 0 \Rightarrow r = \cancel{0}, \frac{10}{3}$

V': $+++$ on $(0, \frac{10}{3})$, $---$ on $(\frac{10}{3}, 5)$; marks at $0, \frac{10}{3}, 5$

hence $V(r)$ increases on $(0, \frac{10}{3})$ and decreases on $[\frac{10}{3}, 5)$

Then the absolute maximum of $V(r)$ occurs when $r = \frac{10}{3}$, $h = \frac{8}{3}$

Therefore the cylinder with maximal volume has radius $\frac{10}{3}$ and height $\frac{8}{3}$

#47.

$$P(\theta) = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

$$\Rightarrow P'(\theta) = \frac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$$

Hence if $P'(\theta) = 0 \Rightarrow \mu\cos\theta - \sin\theta = 0 \Rightarrow \mu = \tan\theta$

Therefore $P(\theta)$ is minimized when $\tan\theta = \mu$ $(\theta = \tan^{-1}\mu)$.

#55.

Let x be the number of customers and P be the net profits in dollars.

Then for $0 \le x \le 250$

$$P(x) = \begin{cases} 12x & 0 \le x \le 50 \\ (12 - 0.06(x-50))x = (15 - 0.06x)x & 50 < x \le 250 \end{cases}$$

$$\Rightarrow P'(x) = \begin{cases} 12 & 0 \le x \le 50 \\ 15 - 0.12x & 50 < x \le 250 \end{cases}$$

$P'(x) = 0$ when $x = \frac{15}{0.12} = 125$

The critical points : $x = 125$ and $x = 50$

And $P(50) = 600$, $P(125) = 937.5$, $P(0) = 0$, $P(250) = 0$

We can conclude that the net profit is maximized by serving 125 customers.

HW12

4. §4.8

#26. $f(x)=\frac{1}{x^3-x}=\frac{1}{x(x^2-1)}$

(1) Domain: The domain of f consists of all $x\neq 0,1,-1$, the set $\mathbb{R}\setminus\{0,1,-1\}$.

The y-axis is a vertical asymptote: $f(x)\to\infty$ as $x\to 0^-$
$f(x)\to-\infty$ as $x\to 0^+$

The line $x=1$ is a vertical asymptote: $f(x)\to\infty$ as $x\to 1^+$
$f(x)\to-\infty$ as $x\to 1^-$

The line $x=-1$ is a vertical asymptote: $f(x)\to\infty$ as $x\to -1^+$
$f(x)\to-\infty$ as $x\to -1$.

The x-axis is a horizontal asymptote: $f(x)\to 0$ as $x\to-\infty$
$f(x)\to 0$ as $x\to\infty$.

(2) Intercepts: There's no y-intercept since f is not defined at $x=0$.
There's no x-intercepts since $f(x)\neq 0$ $\forall x\in\mathbb{R}\setminus\{0,1,-1\}$.

(3) Symmetric: $f(-x)=\frac{1}{(-x)^3-(-x)}=\frac{1}{-x^3+x}=-\frac{1}{x^3-x}=-f(x)$

$\Rightarrow$ The graph is symmetric about the origin

f is NOT periodic.

(4) First derivate: $f(x)=x^{-1}(x^2-1)^{-1}\Rightarrow f'(x)=-x^{-2}(x^2-1)^{-1}-2x\cdot x^{-1}(x^2-1)^{-2}$
$=\frac{-(x^2-1)-2x^2}{x^2(x^2-1)^2}=\frac{-3x^2+1}{x^2(x^2-1)^2}$.

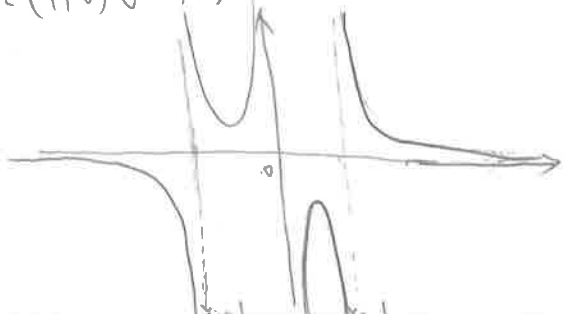
The critical points of f are $x=\pm\frac{1}{\sqrt{3}}$ $\rightarrow$

−	+	−
decreasing $-\frac{1}{\sqrt{3}}$	increasing $\frac{1}{\sqrt{3}}$	decreasing.

(5) Second derivate: $f''(x)=\frac{-6x(x^2-1)^2x^2-(-3x^2+1)2(x^3-x)(3x^2-1)}{x^4(x^2-1)^4}=\frac{-2(-6x^4+3x^2-1)}{(x^3-x)^3}$

Concave up on: $(-1,0)\cup(1,\infty)$ Concave down on $(-\infty,-1)\cup(0,1)$.

Hence the graph:



#37. $f(x) = 2\sin^3 x + 3\sin x \qquad x \in [0, \pi]$

(1) Domain : $[0, \pi]$

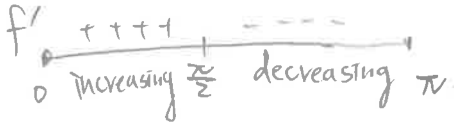
(2) No x-intercepts and y-intercepts.

(3) $f'(x) = 2\sin^2 x\cos x \; 3 \; + 3\cos x$

$= 6\sin^2 x\cos x + 3\cos x = 3\cos x\,(2\sin^2 x + 1)$

The critical points: $x = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = 5$



(4) $f''(x) = -3\sin x(\overset{1-\cos^2 x}{2\sin^2 x} + 1) + 3\cos x\,(2\sin x\cos x \cdot 2)$

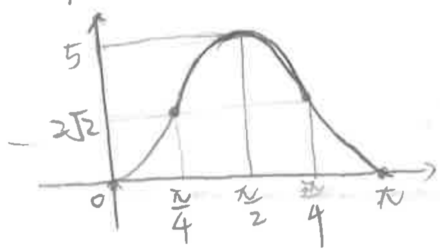
$= -6\sin^3 x - 3\sin x + 12\overset{1-\sin^2 x}{\cos^2 x}\sin x$

$= -6\sin^3 x - 3\sin x + 12\sin x - 12\sin^3 x$

$= 9\sin x - 18\sin^3 x = 9\sin x\,(1 - 2\sin^2 x)$

The points of reflection: $x = \frac{\pi}{4}, \frac{3\pi}{4}$ $\qquad f(\frac{\pi}{4}) = f(\frac{3}{4}\pi) = 2\sqrt{2}$.

Hence the graph:



#51. $f(x) = 2\tan x - \sec^2 x \qquad x \in (0, \frac{\pi}{2})$

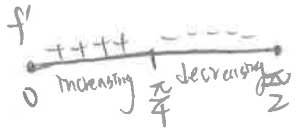
(1) Domain : $(0, \frac{\pi}{2})$

The vertical asymptote : $x = \frac{\pi}{2}$, $\quad f(x) \to -\infty$ as $x \to \frac{1}{2}^-$

(2) No x-intercepts and y-intercepts.

(3) $f'(x) = 2\sec^2 x - 2\sec^2 x\tan x = 2\sec^2 x\,(1 - \tan x)$.

The critical point : $x = \frac{\pi}{4}$. $\qquad f(\frac{\pi}{4}) = 2 - 2 = 0$



(4) $f''(x) = 2\sec^2 x\,(3\tan^2 x - 2\tan x + 1) < 0 \quad \forall\, x \in (0, \frac{\pi}{2})$

No point of reflection.

Hence the graph:

$f(0) = -1$