Hw 13

1、Rudin Ch 6 ※2.

Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)\,dx = 0$.

Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Ex 1)

<pf>

Suppose $f(x_0) > 0$ for some $x_0 \in [a,b]$.

Since f is continuous at x_0, given $\varepsilon = \frac{f(x_0)}{2} > 0$, then

$\exists \delta > 0$ such that if $y \in (x_0 - \delta, x_0 + \delta)$, we have $|f(y) - f(x_0)| < \varepsilon$.

$$\Rightarrow -\varepsilon + f(x_0) < f(y) < \varepsilon + f(x_0)$$

$$\Rightarrow 0 < \frac{f(x_0)}{2} < f(y) < \frac{3f(x_0)}{2}.$$

Since f is continuous on $[a,b]$, then f is Riemannian integrable on $[a,b]$.
$(f \in R([a,b]))$

$$\Rightarrow 0 = \int_a^b f(x)\,dx = \underbrace{\int_a^{x_0-\delta} f(x)\,dx}_{\geq 0 \ (\because f \geq 0)} + \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx + \underbrace{\int_{x_0+\delta}^{b} f(x)\,dx}_{\geq 0 \ (\because f \geq 0)}$$

$$\geq \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx > 0 \quad \text{(contradiction)}$$

So $f(x) = 0$ for all $x \in [a,b]$. ■

2. Rudin, Ch6 #5

Suppose f is a bounded real function on $[a,b]$, and $f^2 \in R$ on $[a,b]$.

Does it follow that $f \in R$? Does the answer change if we assume that $f^3 \in R$?

<Ex>

Define: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [a,b], \\ -1, & x \in \mathbb{Q}^c \cap [a,b]. \end{cases}$ Then $f^2(x) \equiv 1$ on $[a,b]$.

Thus $\int_a^b f^2(x)\,dx = \int_a^b 1\,dx = b-a < \infty$, that is, $f^2 \in R$.

But $\int_a^b f(x)\,dx$ does not exist, that is, $f \notin R$.

<pf>

Since f is a bounded function on $[a,b]$, then f^3 is also bounded function on $[a,b]$.

Let $\phi(x) = \sqrt[3]{x}$ be defined on $\mathbb{R}$.

Since $f^3 \in R$; f^3 is bounded on $[a,b]$, $\phi(x) = \sqrt[3]{x}$ is continuous on $f^3([a,b])$.

by Theorem 6.11, then $\phi \circ f^3 = \sqrt[3]{f^3} = f$ is also Riemannian integral on $[a,b]$,

that is, $f \in R$.

(回到上述例子)

為何 $f^2 \in R$，但不取 $\phi(x)=\sqrt{x}$ is continuous on $[0,\infty)$，再利用 Theorem 6.11 去解釋？

$\because$ $\phi \circ f^2 = \phi(f^2) = \sqrt{f^2} = |f|$，by theorem 6.11

$\therefore$ $|f|$ is Riemannian integral on $[a,b]$，that is，$\int_a^b |f(x)|\,dx < \infty$.

但是"不保證" f is Riemannian integral on $[a,b]$. ※

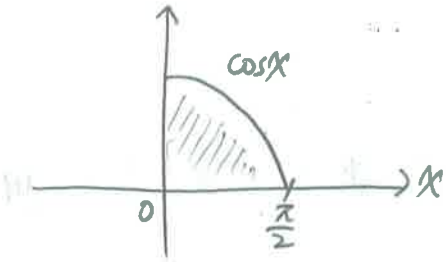
(回想我給的例子!!)

3. Prove, straight from definition, that $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \, dx = 1$

<pf> Since $\cos x$ is monotonically decreasing on $[0, \frac{\pi}{2}]$, then $\int_0^{\frac{\pi}{2}} \cos x \, dx < \infty$.

Let $n \in \mathbb{N}$, and $P_n = \{ 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \frac{\pi}{2} \}$, where $x_i = \frac{i\pi}{2n}$

for all $i = 0, 1, 2, \cdots, n$.



Since $\cos x$ is decreasing on $[0, \frac{\pi}{2}]$,

then $m_i = \inf \{ \cos x \mid x_{i-1} \le x \le x_i \} = \cos(x_i) = \cos\left(\frac{i\pi}{2n}\right)$, $1 \le i \le n$

and $M_i = \sup \{ \cos x \mid x_{i-1} \le x \le x_i \} = \cos(x_{i-1}) = \cos\left(\frac{(i-1)\pi}{2n}\right)$, $1 \le i \le n$.

Now,

$$U(f, P_n) = \sum_{i=1}^{n} \cos\left(\frac{(i-1)\pi}{2n}\right) \cdot \frac{\pi}{2n} = \frac{\pi}{2n} \left[1 + \cos\left(\frac{\pi}{2n}\right) + \cos\left(\frac{2\pi}{2n}\right) + \cdots + \cos\left(\frac{n-1}{2n} \cdot \pi\right) \right] \text{—①}$$

$$L(f, P_n) = \sum_{i=1}^{n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n} = \frac{\pi}{2n} \cdot \left[\cos\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n} \cdot 2\right) + \cdots + \cos\left(\frac{\pi}{2n} \cdot n\right) \right] \text{—②}$$

Formula:

$$\cos x + \cos 2x + \cdots + \cos(nx) = \frac{\sin(n+\frac{1}{2})x - \sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

利用 $e^{i\theta} = \cos\theta + i\sin\theta$ 去整理 $\sum_{i=1}^{n} e^{i\theta}$ 的式子!

$$① \Rightarrow U(f, P_n) = \frac{\pi}{2n} \left[1 + \frac{\sin(n-\frac{1}{2}) \cdot \frac{\pi}{2n} - \sin(\frac{1}{2} \cdot \frac{\pi}{2n})}{2\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} \right]$$

$$② \Rightarrow L(f, P_n) = \frac{\pi}{2n} \left[\frac{\sin(n+\frac{1}{2}) \cdot \frac{\pi}{2n} - \sin(\frac{1}{2} \cdot \frac{\pi}{2n})}{2\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} \right]$$

$$U(f,P_n) = \frac{\pi}{2n}\left[1 + \frac{\sin(n-\frac{1}{2})\cdot\frac{\pi}{2n} - \sin(\frac{1}{2}\cdot\frac{\pi}{2n})}{2\cdot\sin(\frac{1}{2}\cdot\frac{\pi}{2n})}\right]$$

$$= \frac{\pi}{2n}\left[1 + \frac{\sin(\frac{\pi}{2}-\frac{\pi}{4n}) - \sin(\frac{\pi}{4n})}{2\cdot\sin(\frac{\pi}{4n})}\right]$$

$$= \frac{\pi}{2n}\left[1 + \frac{\cos\frac{\pi}{4n} - \sin\frac{\pi}{4n}}{2\cdot\sin(\frac{\pi}{4n})}\right]$$

$$= \frac{\pi}{4n}\cdot\frac{\cos(\frac{\pi}{4n}) + \sin(\frac{\pi}{4n})}{\sin(\frac{\pi}{4n})}$$, let $\theta = \frac{\pi}{4n}$, as $n\to\infty \Rightarrow \theta\to 0$.

$$= \frac{\theta\cos\theta + \theta\sin\theta}{\sin\theta}$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{\theta\to 0}\frac{\theta\cos\theta+\theta\sin\theta}{\sin\theta} \overset{(\frac{0}{0})}{=} \lim_{\theta\to 0}\frac{\cos\theta - \theta\sin\theta + \sin\theta + \theta\cos\theta}{\cos\theta} = \frac{1}{1} = 1.$$

$$L(f,P_n) = \frac{\pi}{2n}\cdot\left[\frac{\sin(n+\frac{1}{2})\cdot\frac{\pi}{2n} - \sin(\frac{1}{2}\cdot\frac{\pi}{2n})}{2\sin(\frac{1}{2}\cdot\frac{\pi}{2n})}\right]$$

$$= \frac{\pi}{4n}\cdot\frac{\sin(\frac{\pi}{2}+\frac{\pi}{4n}) - \sin(\frac{\pi}{4n})}{\sin(\frac{\pi}{4n})} = \frac{\pi}{4n}\cdot\frac{\cos\frac{\pi}{4n} - \sin\frac{\pi}{4n}}{\sin(\frac{\pi}{4n})}$$, let $\theta = \frac{\pi}{4n}$, as $n\to\infty \Rightarrow \theta\to 0$.

$$\lim_{n\to\infty} L(f,P_n) = \lim_{\theta\to 0}\frac{\theta\cos\theta - \theta\sin\theta}{\sin\theta} \overset{(\frac{0}{0})}{=} \lim_{\theta\to 0}\frac{\cos\theta - \theta\sin\theta - \sin\theta - \theta\cos\theta}{\cos\theta} = \frac{1}{1} = 1.$$

Thus $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = 1$,

So $\int_0^{\frac{\pi}{2}} \cos x\, dx = 1$.

(推導#3用的公式): Since $e^{i\theta} = \cos\theta + i\sin\theta,\ \theta \in \mathbb{R}$ (Euler formula)

$$e^{i2\theta} = \cos(2\theta) + i\sin(2\theta) \quad \leftarrow \text{棣美弗定理應用!!}$$

$$e^{i3\theta} = \cos(3\theta) + i\sin(3\theta)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{N},$$

then we have

$$\sum_{k=1}^{n} e^{ik\theta} = \left(\sum_{k=1}^{n} \cos(k\theta)\right) + i\left(\sum_{k=1}^{n} \sin(k\theta)\right).$$

$$\sum_{k=1}^{n} e^{ik\theta} = e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}$$

(公比為 $e^{i\theta}$ 的等比級數總和)

$$\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}} = \frac{(\cos\theta + i\sin\theta)(1-\cos(n\theta) - i\sin(n\theta))}{1-\cos\theta - i\sin\theta} \quad \text{—— ①}$$

$$1-\cos\theta = 1-(1-2\sin^2\tfrac{\theta}{2}) = 2\sin^2\tfrac{\theta}{2} \quad \text{—— ②}$$

$$1-\cos(n\theta) = 1-(1-2\sin^2(\tfrac{n\theta}{2})) = 2\sin^2(\tfrac{n\theta}{2}) \quad \text{—— ③}$$

②+③ 代入 ① $\Rightarrow$

$$\frac{(\cos\theta + i\sin\theta)\cdot[2\sin^2(\tfrac{n\theta}{2}) - i\sin(n\theta)]}{2\sin^2\tfrac{\theta}{2} - i\sin\theta}$$

$$= \frac{(\cos\theta + i\sin\theta)\cdot 2\sin(\tfrac{n\theta}{2})\cdot[\sin(\tfrac{n\theta}{2}) - i\cos(\tfrac{n\theta}{2})]\cdot[\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2}]}{2\sin\tfrac{\theta}{2}\cdot[\sin\tfrac{\theta}{2} - i\cos\tfrac{\theta}{2}]\cdot[\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2}]}$$

$$= \frac{(\cos\theta + i\sin\theta)\cdot 2\sin(\frac{n\theta}{2})\cdot\left[\sin(\frac{n\theta}{2}) - i\cos(\frac{n\theta}{2})\right]\cdot\left[\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}} \quad \text{—④}$$

$$\underline{(\cos\theta + i\sin\theta)\cdot(\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2})} = (\cos\theta\sin\tfrac{\theta}{2} - \sin\theta\cos\tfrac{\theta}{2}) + i(\cos\theta\cdot\cos\tfrac{\theta}{2} + \sin\theta\cdot\sin\tfrac{\theta}{2})$$

$$= \sin(\tfrac{\theta}{2} - \theta) + i\cos(\tfrac{\theta}{2} - \theta) = \sin(\tfrac{-\theta}{2}) + i\cos(\tfrac{-\theta}{2})$$

$$= \underline{-\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2}}$$

$$\underline{(-\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2})\cdot(\sin(\tfrac{n\theta}{2}) - i\cos(\tfrac{n\theta}{2}))}$$

$$= -\sin\tfrac{\theta}{2}\sin(\tfrac{n\theta}{2}) + \cos\tfrac{\theta}{2}\cos(\tfrac{n\theta}{2}) + i\left(\sin\tfrac{\theta}{2}\cos(\tfrac{n\theta}{2}) + \cos\tfrac{\theta}{2}\sin(\tfrac{n\theta}{2})\right)$$

$$= \cos(\tfrac{\theta}{2} + \tfrac{n\theta}{2}) + i\sin(\tfrac{\theta}{2} + \tfrac{n\theta}{2}) = \underline{\cos(\tfrac{\theta+n\theta}{2}) + i\sin(\tfrac{\theta+n\theta}{2})} \quad \text{—⑤}$$

⑤代入④

$$\Rightarrow \frac{2\sin(\frac{n\theta}{2})\cdot\left[\cos(\frac{\theta+n\theta}{2}) + i\sin(\frac{\theta+n\theta}{2})\right]}{2\sin\frac{\theta}{2}} = \left(\sum_{k=1}^{n}\cos(k\theta)\right) + i\left(\sum_{k=1}^{n}\sin(k\theta)\right)$$

$$\Rightarrow \sum_{k=1}^{n}\cos(k\theta) = \frac{\sin(\frac{n\theta}{2})\cdot\cos(\frac{\theta+n\theta}{2})}{\sin(\frac{\theta}{2})}, \quad \sum_{k=1}^{n}\sin(k\theta) = \frac{\sin(\frac{n\theta}{2})\cdot\sin(\frac{\theta+n\theta}{2})}{\sin(\frac{\theta}{2})}$$

$$\sin(\tfrac{n\theta}{2})\cdot\cos(\tfrac{\theta+n\theta}{2}) = \tfrac{1}{2}\cdot\left\{\sin(\tfrac{n\theta}{2} + \tfrac{\theta+n\theta}{2}) + \sin(\tfrac{n\theta}{2} - \tfrac{\theta+n\theta}{2})\right\}$$

$$= \tfrac{1}{2}\cdot\left\{\sin(\tfrac{\theta}{2} + n\theta) - \sin(\tfrac{\theta}{2})\right\}$$

$$\Rightarrow \sum_{k=1}^{n}\cos(k\theta) = \frac{\sin((n+\frac{1}{2})\theta) - \sin(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})}, \text{ 同理, } \sum_{k=1}^{n}\sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}$$

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[redacted]

4. Salas §5-2 ※ 5, 9, 11

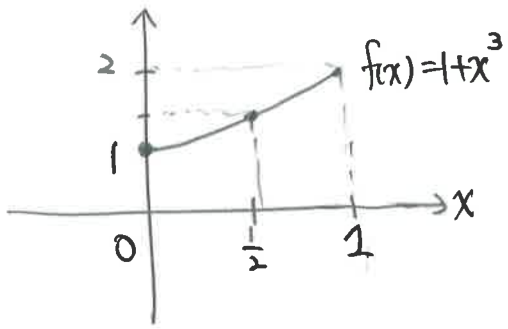
5. $f(x) = 1+x^3$, $x \in [0,1]$, $P = \{0, \frac{1}{2}, 1\}$, Find $L_f(P)$, $U_f(P)$.

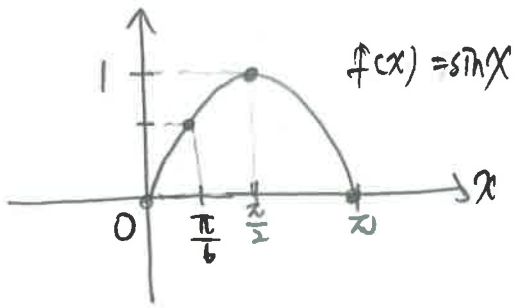
<sol>

$$L_f(P) = f(0)\cdot\frac{1}{2} + f(\tfrac{1}{2})\cdot\frac{1}{2} = 1\times\frac{1}{2} + \frac{9}{8}\times\frac{1}{2} = \frac{17}{16}$$

$$U_f(P) = f(\tfrac{1}{2})\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} = \frac{9}{8}\times\frac{1}{2} + 2\times\frac{1}{2} = \frac{25}{16}$$

※





9. $f(x) = \sin x$, $x \in [0,\pi]$, $P = \{0, \frac{\pi}{6}, \frac{\pi}{2}, \pi\}$ Find $L_f(P)$, $U_f(P)$.

<sol>

$$L_f(P) = f(0)\cdot\frac{\pi}{6} + f(\tfrac{\pi}{6})\cdot\frac{\pi}{3} + f(\pi)\cdot\frac{\pi}{2}$$

$$= 0\times\frac{\pi}{6} + \sin\frac{\pi}{6}\times\frac{\pi}{3} + \sin\pi\times\frac{\pi}{2} = \frac{\pi}{6} + 0\times\frac{\pi}{2} = \frac{\pi}{6}$$

$$U_f(P) = f(\tfrac{\pi}{6})\cdot\frac{\pi}{6} + f(\tfrac{\pi}{2})\cdot\frac{\pi}{3} + \underline{f(\tfrac{\pi}{2})\cdot\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{6}\times\frac{\pi}{6} + \sin\frac{\pi}{2}\times\frac{\pi}{3} + \underline{\sin\frac{\pi}{2}\times\frac{\pi}{2}} = \frac{1}{2}\times\frac{\pi}{6} + \frac{\pi}{3} + \underline{1\times\frac{\pi}{2} = \frac{11\pi}{12}}$$

※

11. f is continuous on $[-1,1]$, P is a partition of $[-1,1]$,

(a) $L_f(P) = 3$, $U_f(P) = 2$

(b) $L_f(P) = 3$, $U_f(P) = 6$, $\int_{-1}^{1} f(x)\,dx = 2$

(c) $L_f(P) = 3$, $U_f(P) = 6$, $\int_{-1}^{1} f(x)\,dx = 10$.

Show that (a), (b), (c) are both false.

<pf>

(a)

Since $L_f(P) \leq U_f(P)$, but $3 \nleq 2$.

(b)

Since $L_f(P) \leq \int_{-1}^{1} f(x)\,dx \leq U_f(P)$, but $3 \nleq 2 \leq 6$.

(c)

Since $L_f(P) \leq \int_{-1}^{1} f(x)\,dx \leq U_f(P)$, but $3 \leq 10 \nleq 6$.

$\blacksquare$