Names and Student IDs: ______________________________

Homework 14 Calculus 1

1. Given $f, f_1, f_2 \in \mathcal{R}([a,b])$, prove that
   - If $f_1(x) \leq f_2(x)\ \forall x \in [a,b]$, then $\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$.
   - $\int_a^b -f(x)dx = -\int_a^b f(x)dx$.
2. - Given $f \in \mathcal{R}([a,b])$ and $c \in [a,b]$, prove that
   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$
   - Prove that if $|f(x)| \leq M\ \forall x \in [a,b]$, then $|\int_a^b f(x)dx| \leq M(b-a)$.
3. Rudin Chapter 6, Problem 8.
4. Suppose that
   $$f(x) = \int_0^x f(t)dt.$$
   - Prove that $f(x) = Ce^x$ for some $C \in \mathbb{R}$.
   - Prove that $f = 0$.
5. Prove the *Cauchy-Schwartz* inequality for integrals:
   $$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx.$$
   (Hint: start by showing that $\int_a^b (f-tg)^2 dx \geq 0$ for $ALL\ t \in \mathbb{R}$.)
6. Let $f(x)$ be *Lipschitz* continuous on $[0,1]$. That is, $\exists M$ such that
   $$|f(x) - f(y)| < M|x-y| \ \ \forall x, y \in [0,1].$$
   Prove that for all $n \in \mathbb{N}$,
   $$\left|\int_0^1 f(x)dx - \frac{1}{n}\sum_{j=1}^n f\left(\frac{j}{n}\right)\right| < \frac{M}{2n},$$
7. Given differentiable functions G, H and continuous function f, prove that
   $$\frac{d}{dx}\int_{H(x)}^{G(x)} f(t)dt = f(G(x))G'(x) - f(H(x))H'(x).$$
8. Salas 5.3: 2, 10, 12, 29, 33.
9. Salas 5.4: 8, 17, 27, 31, 46, 62, 64.