Hw 14

1. Given $f_1, f_2, f \in R([a,b])$, prove that

(1) If $f_1(x) \leq f_2(x)$ $\forall x \in [a,b]$, then $\int_a^b f_1(x)\,dx \leq \int_a^b f_2(x)\,dx$.

(2) $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.

<pf>

(1) Let $P = \{x_0 = a \leq x_1 \leq x_2 \leq \cdots \leq x_n = b\}$ be any partition of $[a,b]$.

Since $f_1(x) \leq f_2(x)$, then $M_i = \sup\limits_{[x_{i-1}, x_i]} [f_2(x) - f_1(x)] \geq 0$ for all $1 \leq i \leq n$,

$m_i = \inf\limits_{[x_{i-1}, x_i]} [f_2(x) - f_1(x)] \geq 0$ for all $1 \leq i \leq n$.

Since $f_1, f_2 \in R([a,b])$, then $f_2 - f_1 \in R([a,b])$.

$$\Rightarrow \int_a^b (f_2(x) - f_1(x))\,dx \geq 0$$

$$\Rightarrow \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f_2(x)\,dx \geq \int_a^b f_1(x)\,dx.$$

$\blacksquare$

(2) Let $P = \{x_0 = a \leq x_1 \leq x_2 \leq \cdots \leq x_n = b\}$ be any partition of $[a,b]$.

$M_i' = \sup\limits_{[x_{i-1}, x_i]} [-f(x)] = -\inf\limits_{[x_{i-1}, x_i]} f(x) = -m_i$ for all $1 \leq i \leq n$

$m_i' = \inf\limits_{[x_{i-1}, x_i]} [-f(x)] = -\sup\limits_{[x_{i-1}, x_i]} f(x) = -M_i$ for all $1 \leq i \leq n$

$$U(P,-f) = \sum_{i=1}^{n} M_i' \Delta x_i = -\sum_{i=1}^{n} m_i \Delta x_i = -L(P,f)$$

$$L(P,-f) = \sum_{i=1}^{n} m_i' \Delta x_i = -\sum_{i=1}^{n} M_i \Delta x_i = -U(P,f)$$

$$\Rightarrow \inf_P U(P,-f) = \inf_P \left(-L(P,f)\right) = -\sup_P L(P,f) = -\underline{\int_a^b} f(x)\,dx$$

$$\underset{\uparrow}{=} -\int_a^b f(x)\,dx \text{ --- } ①$$

$\because f \in R([a,b])$

$$\sup_P L(P,-f) = \sup_P \left(-U(P,f)\right) = -\inf_P U(P,f) = -\overline{\int_a^b} f(x)\,dx$$

$$\underset{\uparrow}{=} -\int_a^b f(x)\,dx \text{ --- } ②$$

$\because f \in R([a,b])$

By ①, ②, then $\inf_P U(P,-f) = \sup_P L(P,-f) = -\int_a^b f(x)\,dx$.

$$\Rightarrow \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

∎

2. (1) Given $f \in R([a,b])$ and $c \in [a,b]$, prove that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(2) Prove that if $|f(x)| \leq M \ \forall x \in [a,b]$, then $\left|\int_a^b f(x)\,dx\right| \leq M \cdot (b-a)$.

<pf>

(1)

① Claim: $f \in R([a,c])$ and $f \in R([c,b])$ $\forall c \in [a,b]$.

given $\varepsilon > 0$, since $f \in R([a,b])$, then $\exists$ partition P of $[a,b]$ such that

$U(f,P) - L(f,P) < \varepsilon$.

Let $P^* = P \cup \{c\} = \{x_0 = a \leq x_1 \leq \cdots \leq x_m = c \leq \cdots \leq x_n = b\}$. $(\Rightarrow P \subseteq P^*)$

Then $U(P^*,f) - L(P^*,f) \leq U(P,f) - L(P,f) < \varepsilon$.

$\Rightarrow U(P^*,f) - L(P^*,f) = \sum_{i=1}^{m} (M_i - m_i)\Delta x_i + \sum_{i=m+1}^{n} (M_i - m_i)\Delta x_i < \varepsilon$,

where $M_i = \sup_{[x_{i-1},x_i]} f(x)$, $m_i = \inf_{[x_{i-1},x_i]} f(x)$, and $M_i - m_i \geq 0$ $\forall 1 \leq i \leq n$.

$\Rightarrow \sum_{i=1}^{m} (M_i - m_i)\Delta x_i < \varepsilon$ and $\sum_{i=m+1}^{n} (M_i - m_i)\Delta x_i < \varepsilon$.

Let $P_1 = \{a = x_0 \leq x_1 \leq \cdots \leq x_m = c\}$ and $P_2 = \{c = x_m \leq x_{m+1} \leq \cdots \leq x_n = b\}$.

Then $U(f,P_1) - L(f,P_1) = \sum_{i=1}^{m} (M_i - m_i)\Delta x_i < \varepsilon$

and $U(f,P_2) - L(f,P_2) = \sum_{i=m+1}^{n} (M_i - m_i)\Delta x_i < \varepsilon$.

So $f \in R([a,c])$ and $f \in R([c,b])$.

② Claim : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $\forall c \in [a,b]$.

given $\varepsilon > 0$, then $\exists$ partition P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$U(P_1, f) \leq \int_a^c f(x)\,dx + \varepsilon/2$ and $U(P_2, f) \leq \int_c^b f(x)\,dx + \varepsilon/2$. (根據 積分定義)

Let $P = P_1 \cup P_2$ be a partition of $[a,b]$. ($\Rightarrow P \leq P_1$ and $P \leq P_2$)

Then $U(P,f) = U(P_1,f) + U(P_2,f) \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon$.

$$\Rightarrow \int_a^b f(x)\,dx \leq U(P,f) \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon.$$

Since ε is arbitrary, then $\int_a^b f(x)\,dx \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

③ Similarly, $\int_a^b f(x)\,dx \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

So $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. ∎

(2) Let $P = \{x_0 = a \leq x_1 \leq \cdots \leq x_n = b\}$ of $[a,b]$.

Since $|f(x)| \leq M$ $\forall x \in [a,b]$, then $-M \leq \inf_{[x_{i-1},x_i]} f(x) = m_i$, $M_i = \sup_{[x_{i-1},x_i]} f(x) \leq M$ $\forall 1 \leq i \leq n$.

$\Rightarrow -M \cdot \Delta x_i \leq m_i \Delta x_i \Rightarrow -M(b-a) = \sum_{i=1}^n M \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i = L(P,f)$

and $M_i \Delta x_i \leq M \cdot \Delta x_i \Rightarrow U(P,f) = \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M \Delta x_i = M(b-a)$.

Since $f \in R([a,b])$, then $-M(b-a) \leq L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f) \leq M(b-a)$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq M \cdot (b-a)$$

∎

(補上)

2、(1) 的③ 的證明：

Claim : $\int_a^b f(x)\,dx \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

given $\varepsilon > 0$, then $\exists$ partition P of $[a,b]$ such that $\int_a^b f(x)\,dx + \varepsilon \geq U(P,f)$.

<u>case 1</u>: $c \in P$. Let $P_1 = P \cap [a,c]$ and $P_2 = P \cap [c,b]$.

Then P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$.

Since $U(P,f) = U(P_1,f) + U(P_2,f)$

$\geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (根據積分定義),

then $\int_a^b f(x)\,dx + \varepsilon \geq U(P,f) \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\Rightarrow \int_a^b f(x)\,dx + \varepsilon \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

<u>case 2</u>: $c \notin P$. Let $P_3 = (P \cap [a,c]) \cup \{c\}$ and $P_4 = (P \cap [c,b]) \cup \{c\}$.

Then P_3 is a partition of $[a,c]$ and P_4 is a partition of $[c,b]$.

Since $P_3 \cup P_4 = P \cup \{c\} \supset P$, then $U(P,f) \geq U(P_3 \cup P_4, f)$.

$\Rightarrow U(P,f) \geq U(P_3 \cup P_4, f) = U(P_3,f) + U(P_4,f)$

$\geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (根據積分定義)

$\Rightarrow \int_a^b f(x)\,dx + \varepsilon \geq U(P,f) \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\Rightarrow \int_a^b f(x)\,dx + \varepsilon \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Since ε is arbitrary, by case 1, case 2, then $\int_a^b f(x)\,dx \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. ∎

3. Rudin Ch6 #8.

Suppose $f \in \mathcal{R}$ on $[a,b]$ for every $a<b$ where a is fixed.

Define $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ if this limit exists (and is finite).

In that case, we say that the integral on the left converges.
If it also converges after f has been replaced by $|f|$, it is said to converge absolutely.

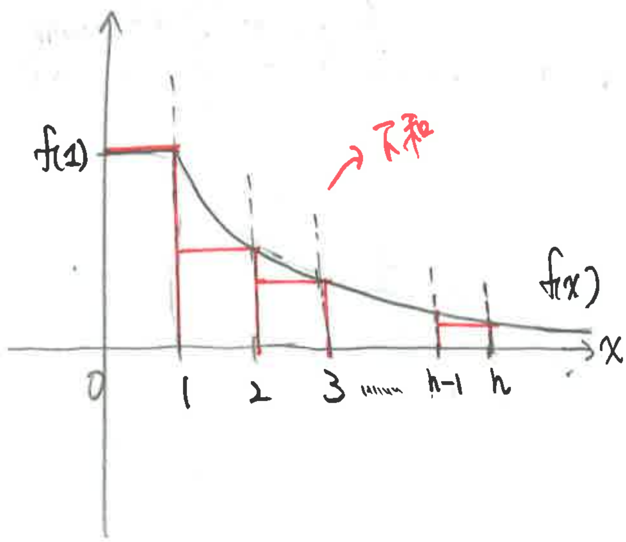
Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1,\infty)$.
Prove that $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.
(This is the so-called "integral test" for convergence of series.)

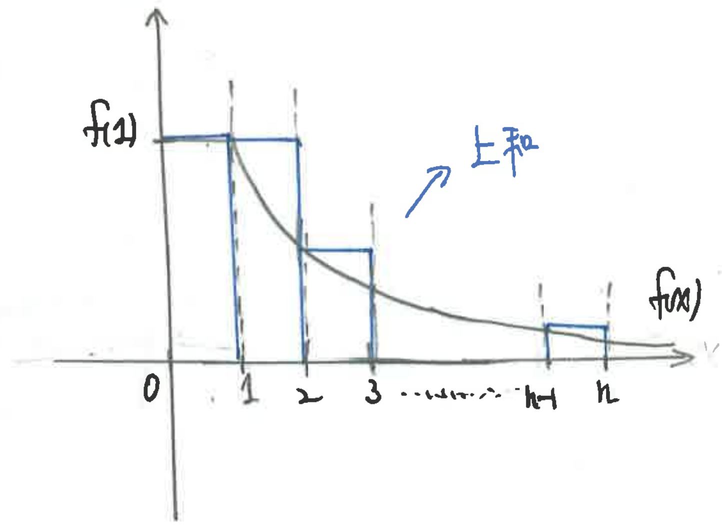
<pf>

For each $n \in \mathbb{N}$, $n>1$, let $\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx$ converges, if the limit exists and is finite.

Define: $f(x) = f(1)$, for any $x \in [0,1]$.

Clearly, $f(x) \geq 0$, $f(x)$ is decreasing monotonically on $[0,\infty)$





By graph, then we have for each $n \in \mathbb{N}$,

(上和)

(下和)

$$f(1)+f(2)+\cdots+f(n-1)+f(n) \le \int_0^n f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^n f(x)\,dx \le f(0)+f(1)+f(2)+\cdots+f(n-1).$$

$$\Rightarrow \sum_{k=1}^{n} f(k) \le f(0) + \int_1^n f(x)\,dx \le f(0)+f(1)+f(2)+\cdots\cdots+f(n-1).$$

$$\Rightarrow -f(0) + \sum_{k=1}^{n} f(k) \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} f(k) \le \sum_{k=1}^{n} f(k) \quad \text{—— } (*)$$

① Since $f \ge 0$ on $[1,\infty)$, then $\sum_{k=1}^{n} f(k)$ and $\int_1^n f(x)\,dx$ are increasing as n is increasing and $\sum_{k=1}^{n} f(k) \ge 0$, $\int_1^n f(x)\,dx \ge 0$ for each $n \in \mathbb{N}$.

② $(\Rightarrow)$ Suppose $\int_1^\infty f(x)\,dx$ converges, that is, $\int_1^\infty f(x)\,dx = \lim_{n\to\infty}\int_1^n f(x)\,dx$ converges.

Since $\lim_{n\to\infty}\int_1^n f(x)\,dx$ converges, then $\exists M>0$ such that $0 \le \int_1^n f(x)\,dx \le M$ $\forall n \in \mathbb{N}$.

By $(*)$, then $0 \le \sum_{k=1}^{n} f(k) \le M + f(0)$ for each $n \in \mathbb{N}$.

Since $\left\{\sum_{k=1}^{n} f(k)\right\}_{n=1}^{\infty}$ is increasing and bounded, then $\sum_{k=1}^{\infty} f(k) = \lim_{n\to\infty}\sum_{k=1}^{n} f(k)$ converges.

③ $(\Leftarrow)$ Suppose $\sum_{k=1}^{\infty} f(k)$ converges, that is, $\sum_{k=1}^{\infty} f(k) = \lim_{n\to\infty}\sum_{k=1}^{n} f(k)$ converges.

Since $\lim_{n\to\infty}\sum_{k=1}^{n} f(k)$ converges, then $\exists M>0$ such that $0 \le \sum_{k=1}^{n} f(k) \le M$ $\forall n \in \mathbb{N}$.

By $(*)$, then $0 \le \int_1^n f(x)\,dx \le M$ for each $n \in \mathbb{N}$.

Since $\left\{\int_1^n f(x)\,dx\right\}_{n=1}^{\infty}$ is increasing and bounded, then $\int_1^\infty f(x)\,dx = \lim_{n\to\infty}\int_1^n f(x)\,dx$ converges.

By ②, ③, we complete this proof. ■

4. Suppose that $f(x) = \int_0^x f(t)\,dt$.

(1) Prove that $f(x) = C \cdot e^x$ for some $C \in \mathbb{R}$.

(2) Prove that $f = 0$.

<pf>

(1)

If f is continuous, then $f'(x) = f(x)$.

$$\frac{df(x)}{dx} = f(x)$$

$$\Rightarrow \frac{df(x)}{f(x)} = dx \Rightarrow \int \frac{df(x)}{f(x)} = \int dx \Rightarrow \ln|f(x)| = x + C, \quad C \text{ is constant.}$$

$$\Rightarrow |f(x)| = e^{x+C} = e^C \cdot e^x \Rightarrow f(x) = \pm e^C \cdot e^x, \text{ let } C_1 = \pm e^C \in \mathbb{R}.$$

$$\Rightarrow f(x) = C_1 \cdot e^x, \quad C_1 \in \mathbb{R}.$$

(2)

Since $f(0) = C_1 \cdot e^0 = C_1$ and $f(0) = \int_0^0 f(t)\,dt = 0$,

then $C_1 = 0$.

So $f(x) = 0$.

5. Prove the Cauchy – Schwartz inequality for integrals:

$$\left[\int_a^b f(x)\cdot g(x)\,dx\right]^2 \leq \left(\int_a^b [f(x)]^2 dx\right)\cdot\left(\int_a^b [g(x)]^2 dx\right).$$

Hint: start by showing that $\int_a^b (f-tg)^2 dx \geq 0$ for all $t \in \mathbb{R}$.

<pf> Case 1: $\int_a^b [f(x)]^2 dx = +\infty$ or $\int_a^b [g(x)]^2 dx = +\infty$, since $f^2 \geq 0$, $g^2 \geq 0$.

Then this inequality holds.

Case 2: $\int_a^b [f(x)]^2 dx < +\infty$ and $\int_a^b [g(x)]^2 dx < +\infty$.

Then we have

$$\int_a^b (f-tg)^2 dx = \int_a^b f^2 dx - 2t\int_a^b fg\,dx + t^2\int_a^b g^2 dx \geq 0$$

(1) If $\int_a^b g^2 dx = 0$, then $g^2 = 0$ on $[a,b] \Rightarrow g = 0$ on $[a,b] \Rightarrow fg = 0$ on $[a,b]$

$\Rightarrow \int_a^b fg\,dx = 0$. So this inequality also holds.

(2) If $\int_a^b g^2 dx > 0$, then we have $\left(-2\cdot\int_a^b fg\,dx\right)^2 - 4\cdot\int_a^b f^2 dx \cdot \int_a^b g^2 dx \leq 0$

$$\Rightarrow \left(\int_a^b fg\,dx\right)^2 \leq \left(\int_a^b f^2 dx\right)\cdot\left(\int_a^b g^2 dx\right).$$

∎

6. Let $f(x)$ be Lipschitz continuous on $[0,1]$. That is, $\exists M$ such that

$$|f(x)-f(y)| \leq M \cdot |x-y| \quad \forall x,y \in [0,1].$$

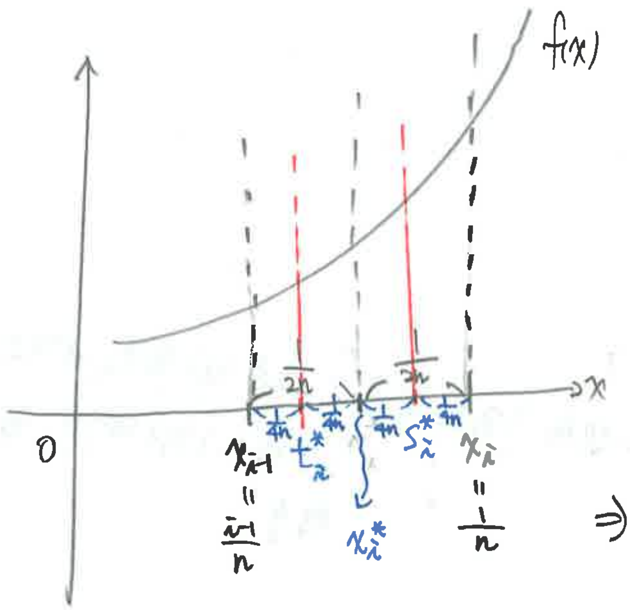
Prove that for all $n \in \mathbb{N}$,

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j}{n}\right) \right| < \frac{M}{2n}.$$

<pf> (老師方法)

Let $P = \{0 = x_0 \leq x_1 \leq \cdots \leq x_n = 1\}$ of $[0,1]$, where $x_i = \frac{i}{n}$, $0 \leq i \leq n$.

(P的組距)



Let $P^* = P \cup \{x_i^*\}$, where $x_i^* = \frac{1}{2}(x_{i-1}+x_i)$, $1 \leq i \leq n$,

$$\left| \frac{1}{n} f(x_i) - \left(\frac{1}{2n} f(x_i^*) + \frac{1}{2n} f(x_i) \right) \right|$$

$$= \left| \frac{1}{2n} f(x_i) - \frac{1}{2n} f(x_i^*) \right|$$

$$\Rightarrow \left| \sum_{i=1}^{n} \frac{1}{n} f(x_i) - \sum_{i=1}^{n} \left(\frac{1}{2n} f(x_i^*) + \frac{1}{2n} f(x_i) \right) \right|$$

$$= \left| \sum_{i=1}^{n} \frac{1}{2n} \left(f(x_i) - f(x_i^*) \right) \right|$$

$$\leq \frac{1}{2n} \sum_{i=1}^{n} |f(x_i) - f(x_i^*)|$$

$$\leq \frac{1}{2n} \cdot M \cdot \sum_{i=1}^{n} |x_i - x_i^*| = \frac{1}{2n} \cdot M \cdot \frac{1}{2n} \cdot n = \frac{1}{2} \cdot \frac{M}{2n}$$

(P*的組距)

Let $P^{**} = P^* \cup \{t_i^*\} \cup \{s_i^*\}$, where $t_i^* = \frac{1}{2}(x_{i-1} + x_i^*)$ and $s_i^* = \frac{1}{2}(x_i^* + x_i)$, $1 \leq i \leq n$,

Claim: $$\left| \sum_{i=1}^{n} \frac{1}{n} f(x_i) - \sum_{i=1}^{n} \left(\frac{1}{4n} f(t_i^*) + \frac{1}{4n} f(x_i^*) + \frac{1}{4n} f(s_i^*) + \frac{1}{4n} f(x_i) \right) \right| < \left(\frac{1}{2} + \frac{1}{4} \right) \cdot \frac{M}{2n}.$$

$$\left|\frac{1}{n}f(x_i)-\left(\frac{1}{4n}f(t_i^*)+\frac{1}{4n}f(x_i^*)+\frac{1}{4n}f(s_i^*)+\frac{1}{4n}f(x_{i})\right)\right|$$

$$=\left|\frac{1}{n}f(x_i)-\left(\frac{1}{2n}f(x_i^*)+\frac{1}{2n}f(x_i)\right)+\left(\frac{1}{2n}f(x_i^*)+\frac{1}{2n}f(x_i)\right)-\left(\frac{1}{4n}f(t_i^*)+\frac{1}{4n}f(x_i^*)+\frac{1}{4n}f(s_i^*)+\frac{1}{4n}f(x_i)\right)\right|$$

$$=\left|\frac{1}{2n}\left(f(x_i)-f(x_i^*)\right)+\frac{1}{4n}\left(f(x_i^*)-f(t_i^*)\right)+\frac{1}{4n}\left(f(x_i)-f(s_i^*)\right)\right|$$

$$\Rightarrow \left|\sum_{i=1}^{n}\frac{1}{n}f(x_i)-\sum_{i=1}^{n}\left(\frac{1}{4n}f(t_i^*)+\frac{1}{4n}f(x_i^*)+\frac{1}{4n}f(s_i^*)+\frac{1}{4n}f(x_i)\right)\right|$$

$$=\left|\sum_{i=1}^{n}\frac{1}{2n}\left(f(x_i)-f(x_i^*)\right)+\sum_{i=1}^{n}\frac{1}{4n}\left(f(x_i^*)-f(t_i^*)\right)+\sum_{i=1}^{n}\frac{1}{4n}\left(f(x_i)-f(s_i^*)\right)\right|$$

$$\le \frac{1}{2n}\sum_{i=1}^{n}\left|f(x_i)-f(x_i^*)\right|+\frac{1}{4n}\sum_{i=1}^{n}\left|f(x_i^*)-f(t_i^*)\right|+\frac{1}{4n}\cdot\sum_{i=1}^{n}\left|f(x_i)-f(s_i^*)\right|$$

$$\le \frac{1}{2n}\cdot M\cdot\frac{1}{2n}\cdot n+\frac{1}{4n}\cdot M\cdot\frac{1}{4n}\cdot n+\frac{1}{4n}\cdot M\cdot\frac{1}{4n}\cdot n$$

$$=\frac{M}{2n}\cdot\left(\frac{1}{2}+\frac{1}{4}\right).$$

每一組的"分割點"都是「上一組的分割點」再加上「上一組的分割點的中點」的聯集。

現在，執行了 k 次，會有：$P^{\overset{(k個)}{**\cdots*}}=P^{\overset{(k-1個)}{**\cdots*}}\cup\{P^{\overset{(k-1個)}{**\cdots*}}$ 的組距中點$\}$ of $[0,1]$ such that

$$\left|\sum_{i=1}^{n}\frac{1}{n}f(x_i)-\sum_{i=1}^{n}\frac{1}{2^k}\cdot\frac{1}{n}\left(f(a)\right)\right|\le\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}\right)\cdot\frac{M}{2n}=\left[1-\left(\frac{1}{2}\right)^k\right]\cdot\frac{M}{2n},$$

where $a\in P^{\overset{(k個)}{**\cdots*}}$ (取右端點!!)

As $k\to\infty$, then we have

$$\left|\sum_{i=1}^{n}\frac{1}{n}f(x_i)-\int_0^1 f(x)\,dx\right|\le\frac{M}{2n}.$$

[$\because$ f is Lipschitz continuous

$\Rightarrow$ f is continuous

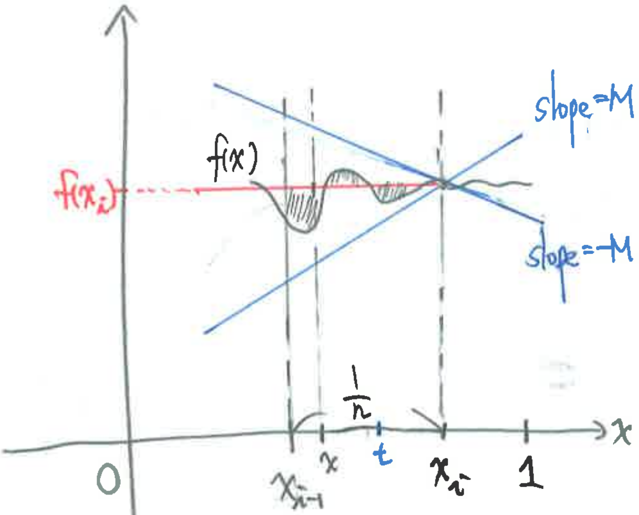
$\Rightarrow$ $f\in R([0,1])$]

∎

(学生做法)

Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0,1]$,

where $x_i = \frac{i}{n}$, $0 \le i \le n$.



Question: For any $1 \le i \le n$, given $x \in (x_{i-1}, x_i)$, Ask

$\left|\int_{x_{i-1}}^{x_i} f(t) - f(x_i)\, dt\right| \le \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{M}{n}$? Answer: Yes.

Since $|f(t) - f(x_i)| \le M\cdot|t - x_i| = M\cdot(x_i - t)$, $t \in (x, x_i)$,

then $-M\cdot(x_i - t) \le f(t) - f(x_i) \le M\cdot(x_i - t)$

$$\Rightarrow -\int_x^{x_i} M\cdot(x_i - t)\,dt \le \int_x^{x_i} f(t) - f(x_i)\,dt \le \int_x^{x_i} M\cdot(x_i - t)\,dt.$$

$$\because \int_x^{x_i} (x_i - t)\,dt = x_i t - \frac{1}{2}t^2\Big|_{t=x}^{t=x_i} = \left(x_i^2 - \frac{1}{2}x_i^2\right) - \left(x_i x - \frac{1}{2}x^2\right)$$

$$= \frac{1}{2}\left(x_i^2 - 2x x_i + x^2\right) = \frac{1}{2}(x_i - x)^2 \le \frac{1}{2}\cdot\frac{1}{n^2}$$

$\therefore -M\cdot\frac{1}{2}\cdot\frac{1}{n^2} \le \int_x^{x_i} f(t) - f(x_i)\,dt \le M\cdot\frac{1}{2}\cdot\frac{1}{n^2}$ for any $1 \le i \le n$, as $x \in (x_{i-1}, x_i)$.

$\Rightarrow \frac{-M}{2n^2} \le \int_{x_{i-1}}^{x_i} f(t) - f(x_i)\,dt \le \frac{M}{2n^2}$ for any $1 \le i \le n$.

Then $\left|\frac{1}{n}\sum_{i=1}^{n} f(x_i) - \int_0^1 f(t)\,dt\right| = \left|\sum_{i=1}^{n}\left(\int_{x_{i-1}}^{x_i} f(x_i)\,dx - \int_{x_{i-1}}^{x_i} f(t)\,dt\right)\right|$

$$\le \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} f(t) - f(x_i)\,dt\right|$$

$$\le \sum_{i=1}^{n}\frac{M}{2n^2} = \frac{M}{2n^2}\times n = \frac{M}{2n}.$$

∎

7. Given differentiable function G, H and continuous function f, prove that

$$\frac{d}{dx}\int_{H(x)}^{G(x)} f(t)\,dt = f(G(x))\cdot G'(x) - f(H(x))\cdot H'(x).$$

<pf>

Given an $a \in \mathbb{R}$ between $G(x)$ and $H(x)$, then

$$\frac{d}{dx}\int_{H(x)}^{G(x)} f(t)\,dt$$

$$= \frac{d}{dx}\left(\int_a^{G(x)} f(t)\,dt - \int_a^{H(x)} f(t)\,dt\right),$$ since f is continuous,

$$= \frac{d}{dx}\left(\int_a^{G(x)} f(t)\,dt\right) - \frac{d}{dt}\left(\int_a^{H(x)} f(t)\,dt\right),$$ since G, H are differentiable,

$$= f(G(x))\cdot G'(x) - f(H(x))\cdot H'(x).$$

Claim: $\frac{d}{dx}\left(\int_a^{G(x)} f(t)\,dt\right) = f(G(x))\cdot G'(x)$ and $\frac{d}{dx}\left(\int_a^{H(x)} f(t)\,dt\right) = f(H(x))\cdot H'(x)$.

Let $F(x) = \int_a^{x} f(t)\,dt$, $u = G(x)$, $v = H(x)$.

Then $\int_a^{G(x)} f(t)\,dt = F(G(x))$ and $\int_a^{H(x)} f(t)\,dt = F(H(x))$.

By chain rule, then $\frac{d}{dx}\left(\int_a^{G(x)} f(t)\,dt\right) = \frac{dF(G(x))}{dx} = \frac{dF(u)}{du}\cdot\frac{du}{dx} = F(G(x))\cdot G'(x)$

and $\frac{d}{dx}\left(\int_a^{H(x)} f(t)\,dt\right) = \frac{dF(H(x))}{dx} = \frac{dF(v)}{dv}\cdot\frac{dv}{dx} = F(H(x))\cdot H'(x).$

$\blacksquare$

8, Hw 14,

Salas §5-3 ※ 2, 10, 12, 29, 33.

2、 Give that $\int_1^4 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = 7$, $\int_1^8 f(x)\,dx = 11$

<sol>

(a)

$$\int_1^8 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^8 f(x)\,dx$$

$$11 = 5 + \int_4^8 f(x)\,dx \quad , \quad \int_4^8 f(x)\,dx = 6$$

(b)

$$\int_4^3 f(x)\,dx = -\int_3^4 f(x)\,dx = -7$$

(c)

$$\int_1^4 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$5 = \int_1^3 f(x)\,dx + 7 \quad , \quad \int_1^3 f(x)\,dx = -2$$

(d)

$$\int_1^8 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^8 f(x)\,dx$$

$$11 = -2 + \int_3^8 f(x)\,dx \quad , \quad \int_3^8 f(x)\,dx = 13$$

(e)

$$\int_1^8 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^8 f(x)\,dx$$

$$11 = 5 + \int_4^8 f(x)\,dx \quad , \quad \int_4^8 f(x)\,dx = 6 \Rightarrow \int_8^4 f(x)\,dx = -6$$

(f)

$$\int_4^4 f(x)\,dx = 0$$

10.

$$F(x) = \int_1^x \sin(\pi t)\,dt$$

<sol>

By F.T.C, then $F'(x) = \sin(\pi x)$.

(a)

$$F'(-1) = \sin(-\pi) = -\sin\pi = 0$$

(b)

$$F'(0) = \sin(0) = 0$$

(c) $F'\left(\frac{1}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

(d)

Since $F'(x) = \sin(\pi x)$, then $F''(x) = \pi\cos(\pi x)$.

12.

$$F(x) = \int_2^x (t+1)^3 \, dt.$$

<sol>

By F.T.C., then $F'(x) = (x+1)^3$.

(a)

$$F'(-1) = (-1+1)^3 = 0$$

(b)

$$F'(0) = (0+1)^3 = 1$$

(c)

$$F'\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}+1\right)^3 = \left(\tfrac{3}{2}\right)^3 = \frac{27}{8}$$

(d)

Since $F'(x) = (x+1)^3$, then $F''(x) = 3(x+1)^2$.

29.

$$F(x) = 2x + \int_0^x \frac{\sin(2t)}{1+t^2} \, dt.$$

<sol>

By F.T.C., then $F'(x) = 2 + \dfrac{\sin(2x)}{1+x^2}$ and

$$F''(x) = \frac{(1+x^2)\cdot\cos(2x)\cdot 2 - \sin(2x)\cdot 2x}{(1+x^2)^2}$$

(a) $F(0) = 0$

(b) $F'(0) = 2$

(c) $F''(0) = 2$

33.

If f is continuous on $[a,b]$, then there is a least one number c in (a,b) for which

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

<pf>

Define: $F(x) = \int_a^x f(t)\,dt$ for any $x \in [a,b]$.

By Theorem 5.3.5, then $F(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$.

By M.V.T., then $F(b) - F(a) = F'(c)\cdot(b-a)$ for some $c \in (a,b)$.

$$\Rightarrow \int_a^b f(t)\,dt - 0 = f(c)\cdot(b-a)$$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)(b-a) \text{ for some } c \in (a,b).$$

$\blacksquare$

9. Salas §5-4 ※ 8, 17, 27, 31, 46, 62, 64

8. $\int_1^2 \left(\frac{3}{x^3}+5x\right) dx = ?$

<sol>

Since $\left(\frac{-3}{2}x^{-2}+\frac{5}{2}x^2\right)' = 3x^{-3}+5x$, then

$$\int_1^2 \left(\frac{3}{x^3}+5x\right) dx = \left(\frac{-3}{2}x^{-2}+\frac{5}{2}x^2\right)\Big|_1^2$$

$$= \left(\frac{-3}{2}\times\frac{1}{4}+\frac{5}{2}\times 4\right) - \left(\frac{-3}{2}+\frac{5}{2}\right) = \frac{69}{8}$$

※

17. $\int_0^a (\sqrt{a}-\sqrt{x})^2 dx = ?$

<sol>

$(\sqrt{a}-\sqrt{x})^2 = a+x-2\sqrt{ax}$, $a>0$

Since $\left(ax+\frac{1}{2}x^2-\frac{4}{3}\sqrt{a}\cdot x^{\frac{3}{2}}\right)' = a+x-2\sqrt{a}\sqrt{x}$, $a>0$,

then $\int_0^a (\sqrt{a}-\sqrt{x})^2 dx = \int_0^a (a+x-2\sqrt{a}\cdot\sqrt{x})\, dx$

$$= \left[ax+\frac{1}{2}x^2-\frac{4}{3}\sqrt{a}\cdot x^{3/2}\right]\Big|_0^a$$

$$= \left(a^2+\frac{1}{2}a^2-\frac{4}{3}\sqrt{a}\cdot a\sqrt{a}\right) - 0$$

$$= a^2+\frac{1}{2}a^2-\frac{4}{3}a^2 = \frac{1}{6}a^2$$

※

27、 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc u \cdot \cot u \, du = ?$

<sol>

Since $(-\csc u)' = \csc u \cdot \cot u$, then

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc u \cdot \cot u \, du = \left[-\csc u\right]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= -\csc\frac{\pi}{4} + \csc\frac{\pi}{6} = 2 - \sqrt{2}$$

31、 $\int_{0}^{\frac{\pi}{3}} \left(\frac{2}{\pi}x - 2\sec^2 x\right) dx = ?$

<sol>

Since $\left(\frac{1}{\pi}x^2 - 2\tan x\right)' = \frac{2x}{\pi} - 2\sec^2 x$, then

$$\int_{0}^{\frac{\pi}{3}} \left(\frac{2x}{\pi} - 2\sec^2 x\right) dx = \left[\frac{x^2}{\pi} - 2\tan x\right]\Big|_{0}^{\frac{\pi}{3}}$$

$$= \frac{1}{\pi}\cdot\left(\frac{\pi}{3}\right)^2 - 2\tan\frac{\pi}{3}$$

$$= \frac{\pi}{9} - 2\sqrt{3}$$

46.

<sol>

(a) $\int_{-4}^{2} (2x+3)\,dx = ?$

Since $(x^2+3x)' = 2x+3$, then $\int_{-4}^{2} (2x+3)\,dx = [x^2+3x]\Big|_{-4}^{2}$

$$= (4+6) - (16-12) = 6$$

(b) $$\int_{-4}^{2} |2x+3|\,dx = \int_{-4}^{\frac{-3}{2}} -(2x+3)\,dx + \int_{-3/2}^{2} (2x+3)\,dx$$

$$= (-x^2-3x)\Big|_{-4}^{\frac{-3}{2}} + (x^2+3x)\Big|_{\frac{-3}{2}}^{2}$$

$$= \frac{25}{4} + \frac{49}{4} = \frac{84}{4} = \frac{37}{2}$$

62. Let f be a function such that f' is continuous on $[a,b]$.

Show that $\int_a^b f(t) f'(t)\,dt = \frac{1}{2}[f^2(b) - f^2(a)]$.

<sol>

Define: $F(x) = f^2(x)$ is defined on $[a,b]$.

Since f' is continuous on $[a,b]$, then f is differentiable on $[a,b]$ ($f'(x)$ exist $\forall x \in [a,b]$.)

$\Rightarrow$ f is continuous on $[a,b]$

$\Rightarrow$ F is continuous on $[a,b]$.

Since $F'(x) = 2f(x) f'(x)$, then $\int_a^b f(t) f'(t)\,dt = \frac{1}{2} F(x)\Big|_a^b$

$$= \frac{1}{2}[F(b) - F(a)].$$

64.

f is a continuous function, $F(x) = \int_0^x x f(t)\, dt$.

Find $F'(x)$.

Hint : The answer is <u>not</u> $x f(x)$.

<sol>

$$F(x) = \int_0^x x f(t)\, dt = x \cdot \int_0^x f(t)\, dt$$

By product rule, then we have

$$F'(x) = (x)' \cdot \int_0^x f(t)dt + x \cdot \left(\int_0^x f(t)\, dt \right)'$$

$$= \int_0^x f(t)dt + x \cdot f(x)$$

$\blacksquare$