Hw 15.

1, Salas §5-7 ※ 2, 10, 16, 24, 40, 76.

2, $$\int \frac{1}{\sqrt{2x+1}}\,dx = \int \frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,du = \sqrt{u}+C = \sqrt{2x+1}+C$$
※

$u=2x+1$

$du = 2dx$

10. $$\int x^{n-1}\cdot\sqrt{a+bx^n}\,dx = \int \frac{1}{nb}\sqrt{u}\,du = \frac{1}{nb}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}+C$$

$u = a+bx^n$ $$= \frac{2}{3}\cdot\frac{1}{nb}\cdot(a+bx^n)^{3/2}+C$$

$du = nbx^{n-1}dx$ ※

$\frac{1}{nb}du = x^{n-1}dx$

16. $$\int 2x^3\cdot(1-x^4)^{\frac{-1}{4}}dx = \int \frac{-1}{2}\cdot u^{\frac{-1}{4}}du = \frac{-1}{2}\cdot\frac{4}{3}\cdot u^{\frac{3}{4}}+C$$

$u=1-x^4$ $$= \frac{-2}{3}\cdot(1-x^4)^{\frac{3}{4}}+C$$

$du = -4x^3dx$ ※

$\frac{-1}{2}du = 2x^3dx$

24.

$$\int_0^3 \frac{r}{\sqrt{r^2+16}}\,dr = \int_{16}^{25} \frac{1}{2}\cdot\frac{1}{\sqrt{u}}\,du = \sqrt{u}\,\Big|_{u=16}^{u=25}$$

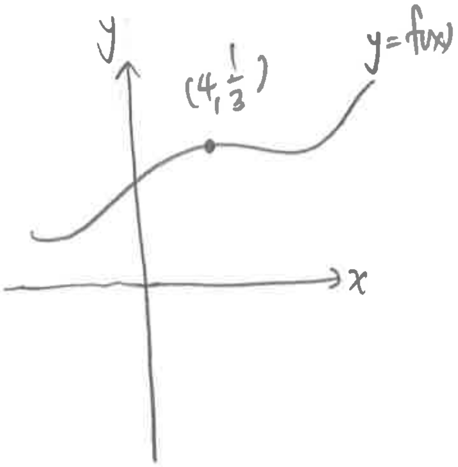
$u = r^2+16 \qquad r=0 \rightarrow u=16 \qquad = \sqrt{25}-\sqrt{16}$

$du = 2r\,dr \qquad r=3 \rightarrow u=25 \qquad = 5-4$

$\frac{1}{2}\,du = r\,dr \qquad\qquad = 1$

40.

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}\,(1+\sqrt{x})^2}$$



$$\Rightarrow \int 1\,dy = \int \frac{-1}{2\sqrt{x}\,(1+\sqrt{x})^2}\,dx$$

$$\Rightarrow y = \int \frac{-1}{2\sqrt{x}\,(1+\sqrt{x})^2}\,dx = \int \frac{-1}{u^2}\,du$$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= \frac{1}{u} + C$$

$$= \frac{1}{1+\sqrt{x}} + C$$

$$\because f(4) = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{1+\sqrt{4}} + C = \frac{1}{3} + C \Rightarrow C = 0$$

$$\therefore f(x) = \frac{1}{1+\sqrt{x}}$$

76.

$$\int_0^{2\pi} \sin^2 x \, dx = \int_0^{2\pi} \frac{1-\cos 2x}{2} dx$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) \Big|_0^{2\pi}$$

$$= \pi$$

※

§5-9 ※26.

<pf>

Claim: $\int_a^b (x-c)\lambda(x)\,dx = 0$ at $c = x_M$ (the center of mass).

$$\int_a^b (x - x_M)\lambda(x)\,dx$$ (See 課本 P287 的解釋)

$= \int_a^b x\cdot\lambda(x)\,dx - \int_a^b x_M\cdot\lambda(x)\,dx$, since $\int_a^b x\cdot\lambda(x)\,dx = x_M\cdot M$,

$\int_a^b \lambda(x)\,dx = M$,

$$= \int_a^b x\cdot\lambda(x)\,dx - x_M\cdot\int_a^b \lambda(x)\,dx$$

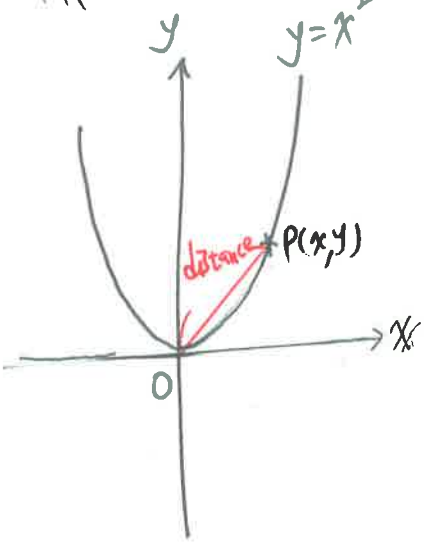
$$= x_M\cdot M - x_M\cdot M$$

$$= 0.$$

■

2、Salas §5-9 ※17、26

17、



distance $= \overline{OP} = \sqrt{x^2+y^2} = \sqrt{x^2+x^4}$

Let $f(x) = \sqrt{x^2+x^4}$, $0 \le x \le \sqrt{3}$.

$$\text{Average} = \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} f(x)\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \sqrt{x^2+x^4}\,dx$$

令 $x = \tan\theta$, $0 \le \theta \le \frac{\pi}{2}$

$dx = \sec^2\theta\,d\theta$

$x=0 \to \theta = 0$

$x=\sqrt{3} \to \theta = \frac{\pi}{3}$

$$= \frac{1}{\sqrt{3}}\cdot\int_0^{\frac{\pi}{3}} \sqrt{\tan^2\theta\,(1+\tan^2\theta)}\cdot\sec^2\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\cdot\int_0^{\frac{\pi}{3}} \tan\theta\cdot\sec^3\theta\,d\theta$$

$u = \sec\theta$, $0 \le \theta \le \frac{\pi}{2}$

$\theta = 0 \to u = 1$

$\theta = \frac{\pi}{3} \to u = 2$

$du = \sec\theta\cdot\tan\theta\,d\theta$

$$= \frac{1}{\sqrt{3}}\int_1^2 u^2\,du$$

$$= \frac{1}{\sqrt{3}}\cdot\left(\frac{1}{3}u^3\Big|_{u=1}^{u=2}\right) = \frac{1}{\sqrt{3}}\cdot\left(\frac{8}{3}-\frac{1}{3}\right)$$

$$= \frac{7}{3\sqrt{3}}$$

$$= \frac{7\sqrt{3}}{9}$$

※

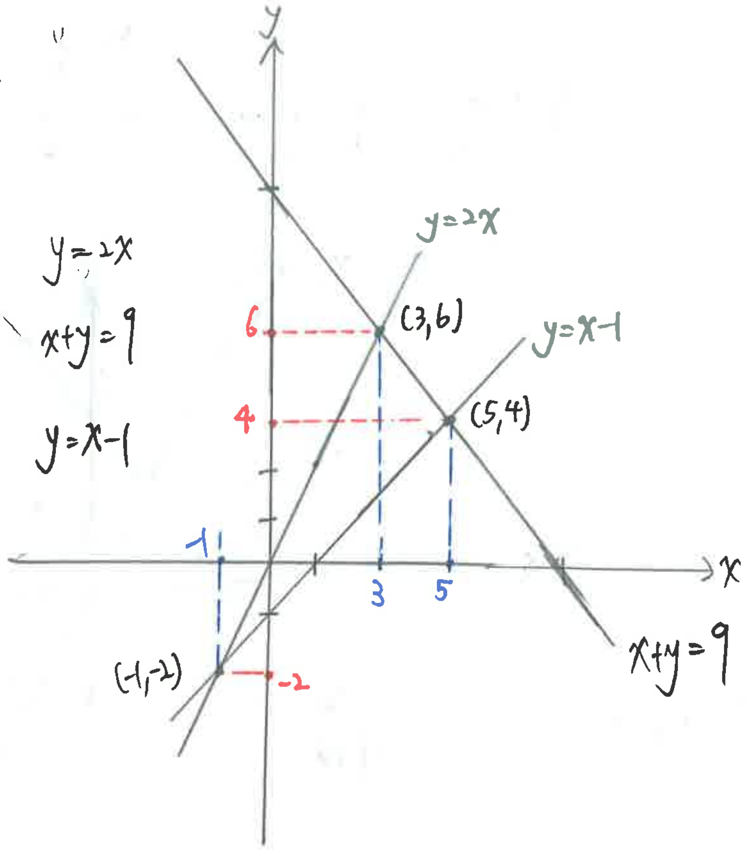
3. Salas §6-1 ※ 11. 18, 22, 34

11.

$y = 2x$

$x + y = 9$

$y = x - 1$



(a) $\int_{-1}^{3} [2x - (x-1)]\,dx + \int_{3}^{5} [(9-x) - (x-1)]\,dx$

$= \int_{-1}^{3} x+1\,dx + \int_{3}^{5} 10 - 2x\,dx = \left(\frac{1}{2}x^2 + x\right)\Big|_{x=-1}^{x=3} + \left(10x - x^2\right)\Big|_{x=3}^{x=5}$

$= 8 + 4 = 12$ ※

(b) $\int_{-2}^{4} \left[(1+y) - \frac{y}{2}\right] dy + \int_{4}^{6} \left[(9-y) - \frac{y}{2}\right] dy$

$= \int_{-2}^{4} 1 + \frac{y}{2}\,dy + \int_{4}^{6} 9 - \frac{3}{2}y\,dy$

$= \left(y + \frac{1}{4}y^2\right)\Big|_{y=-2}^{y=4} + \left(9y - \frac{3}{4}y^2\right)\Big|_{y=4}^{y=6}$

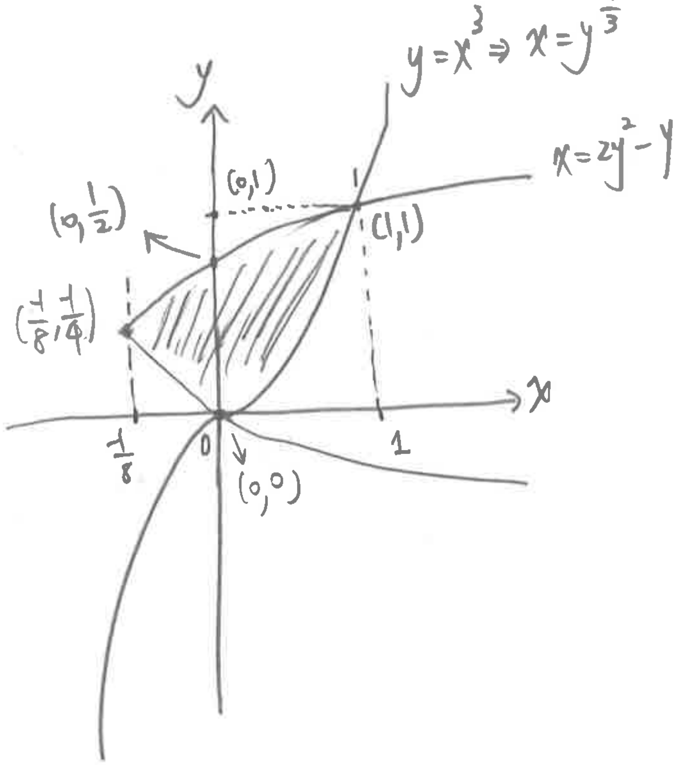
$= 9 + 3 = 12$

※

18.

$x+y=2y^2$, $y=x^3$

$x=2y^2-y=2\left(y^2-\frac{1}{2}y\right)=2\left(y-\frac{1}{4}\right)^2-\frac{1}{8}$ $\therefore V\left(\frac{-1}{8},\frac{-1}{4}\right)$



$$\text{Area}=\int_0^1 y^{\frac{1}{3}}-(2y^2-y)\,dy$$

$$=\int_0^1 y^{\frac{1}{3}}-2y^2+y\,dy$$

$$=\frac{3}{4}y^{\frac{4}{3}}-\frac{2}{3}y^3+\frac{1}{2}y^2\Big|_{y=0}^{y=1}$$

$$=\frac{3}{4}-\frac{2}{3}+\frac{1}{2}=\frac{7}{12}$$

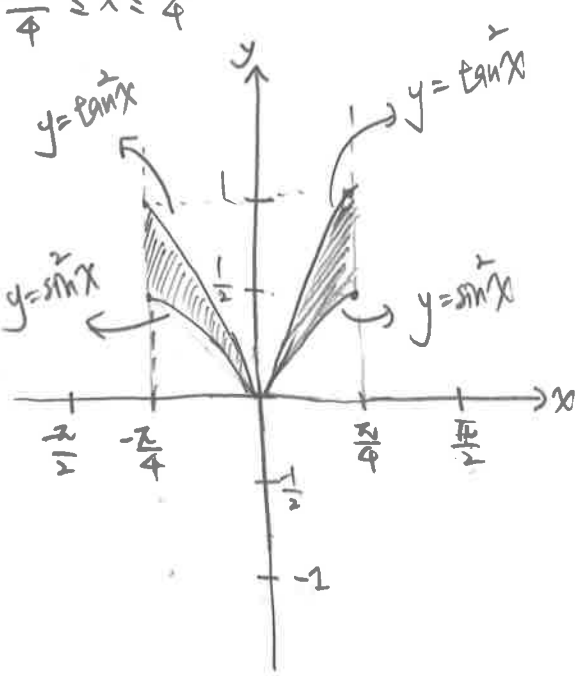
※

22.

$y=\sin^2 x$

$y=\tan^2 x$

$\frac{-\pi}{4}\le x\le\frac{\pi}{4}$



$$\text{Area}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\tan^2x-\sin^2x)\,dx$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\sec^2x-1-\frac{1-\cos 2x}{2}\right)dx$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\sec^2x+\frac{1}{2}\cos 2x-\frac{3}{2}\right)dx$$

$$=\tan x+\frac{1}{4}\sin 2x-\frac{3}{2}x\Big|_{x=\frac{-\pi}{4}}^{x=\frac{\pi}{4}}$$

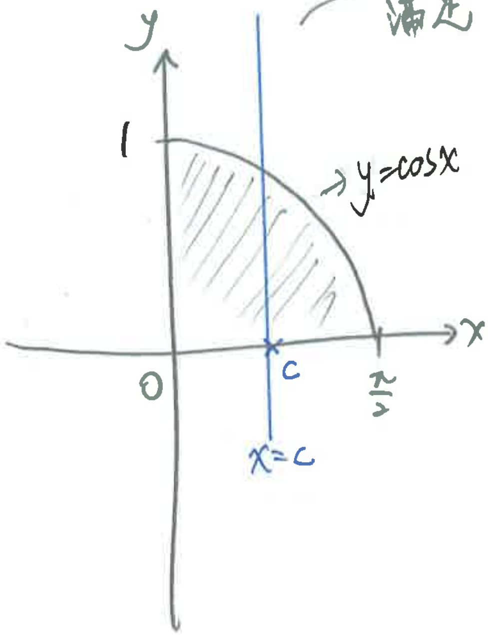
$$=\left(1+\frac{1}{4}-\frac{3\pi}{8}\right)-\left(-1-\frac{1}{4}+\frac{3\pi}{8}\right)$$

$$=\frac{5}{2}-\frac{3\pi}{4}$$

※

34、

$y = \cos x$



满足 $\int_0^c \cos x \, dx = \int_c^{\frac{\pi}{2}} \cos x \, dx$

$\Rightarrow \int_0^c \cos x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos x \, dx$

$\Rightarrow \sin x \Big|_{x=0}^{x=c} = \frac{1}{2} \cdot \left(\sin x \Big|_{x=0}^{x=\frac{\pi}{2}} \right)$

$\Rightarrow \sin c = \frac{1}{2}$

$\Rightarrow c = \frac{\pi}{6}$ ※

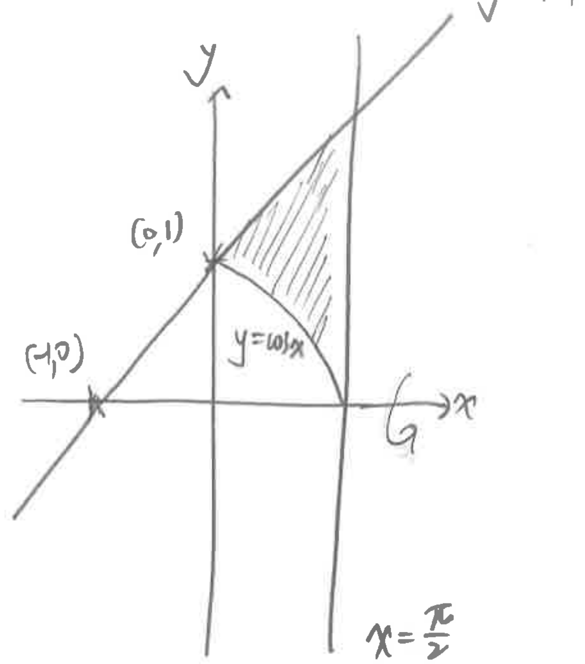
4. Salas §6-2 ※ 15, 24, 35, 43,

15,

$y=\cos x$

$y=x+1$

$x=\frac{\pi}{2}$



$$V=\int_0^{\frac{\pi}{2}}\pi\left[(x+1)^2-\cos^2 x\right]dx$$

$$=\pi\cdot\int_0^{\frac{\pi}{2}}\left[x^2+2x+1-\frac{1+\cos 2x}{2}\right]dx$$

$$=\pi\cdot\int_0^{\frac{\pi}{2}}\left[x^2+2x+\frac{1}{2}-\frac{1}{2}\cos 2x\right]dx$$

$$=\pi\cdot\left[\frac{1}{3}x^3+x^2+\frac{1}{2}x-\frac{1}{4}\sin 2x\Big|_{x=0}^{x=\frac{\pi}{2}}\right]$$

$$=\pi\cdot\left[\frac{\pi^3}{24}+\frac{\pi^2}{4}+\frac{\pi}{4}\right]$$

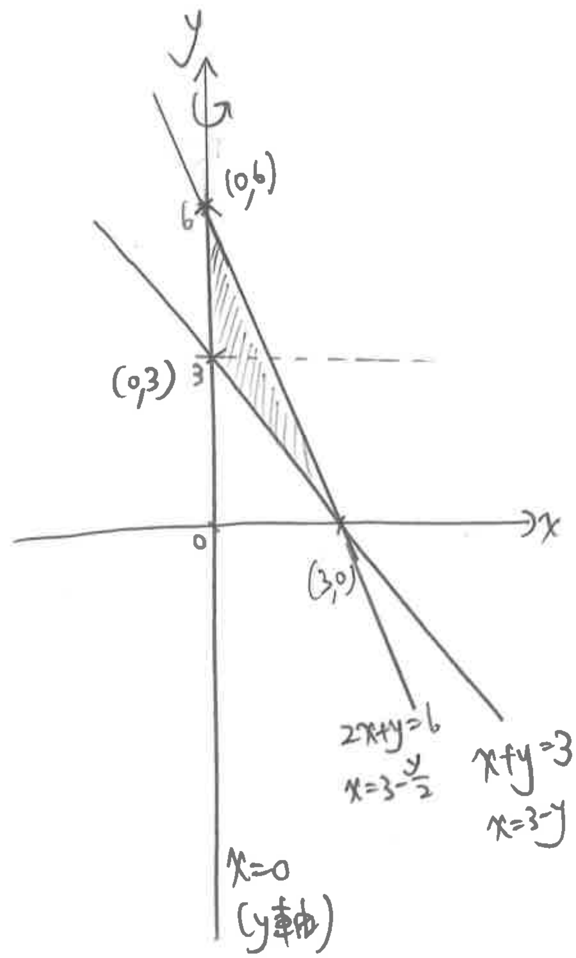
※

24,

$x+y=3$

$2x+y=6$

$x=0$



$$V=\int_0^3\pi\left[(3-\frac{y}{2})^2-(3-y)^2\right]dy+\int_3^6\pi(3-\frac{y}{2})^2dy$$

$$=\pi\cdot\int_0^3 3y-\frac{3}{4}y^2\,dy+\pi\cdot\int_3^6 9-3y+\frac{1}{4}y^2\,dy$$

$$=\pi\cdot\left(\frac{3}{2}y^2-\frac{1}{4}y^3\Big|_{y=0}^{y=3}\right)$$

$$+\pi\cdot\left(9y-\frac{3}{2}y^2+\frac{1}{12}y^3\Big|_{y=3}^{y=6}\right)$$

$$=\pi\cdot\left(\frac{27}{2}-\frac{27}{4}\right)$$

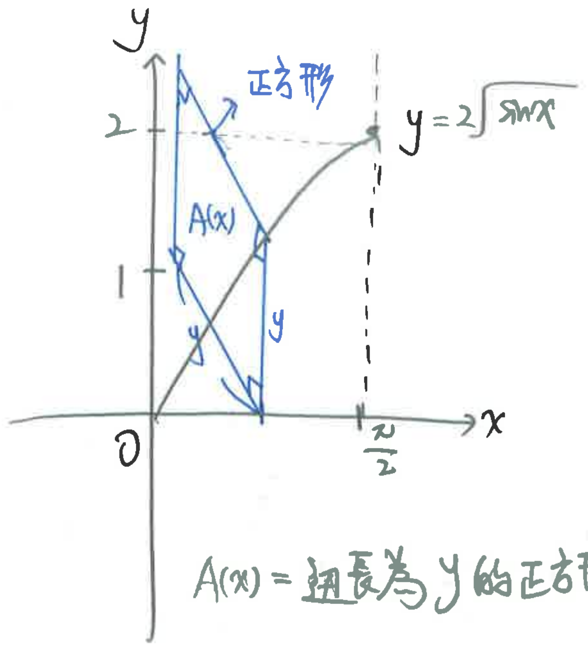
$$+\pi\cdot\left[(54-54+18)-(27-\frac{27}{2}+\frac{9}{4})\right]$$

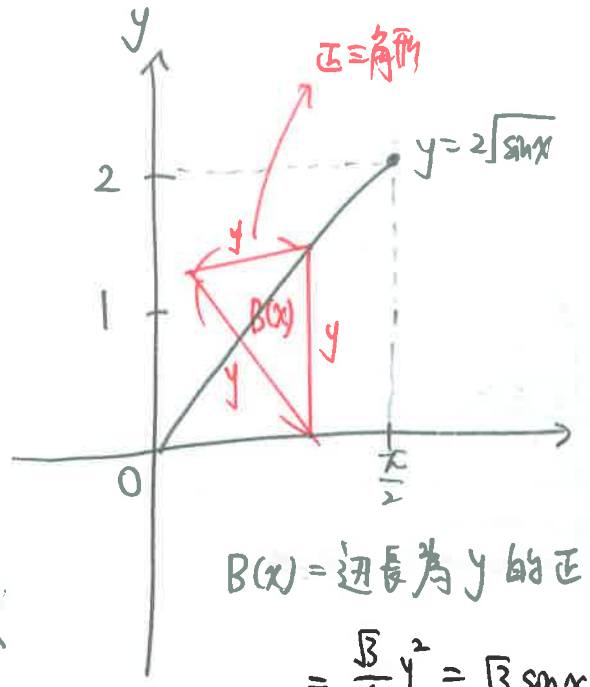
$$=\frac{27}{4}\pi+18\pi-\frac{63}{4}\pi$$

$$=9\pi$$

※

35、 $y=2\sqrt{\sin x}$





A(x) = 边長為 y 的正方形面積

$= y^2 = 4\sin x$

B(x) = 边長為 y 的正三角形面積

$= \frac{\sqrt{3}}{4}y^2 = \sqrt{3}\sin x$

(b)

$$V = \int_0^{\frac{\pi}{2}} 4\sin x \, dx$$

$$= -4\cos x \Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= 0-(-4)$$

$$= 4$$

\#

(a)

$$V = \int_0^{\frac{\pi}{2}} \sqrt{3}\sin x \, dx$$

$$= -\sqrt{3}\cos x \Big|_{x=0}^{x=\frac{\pi}{2}}$$

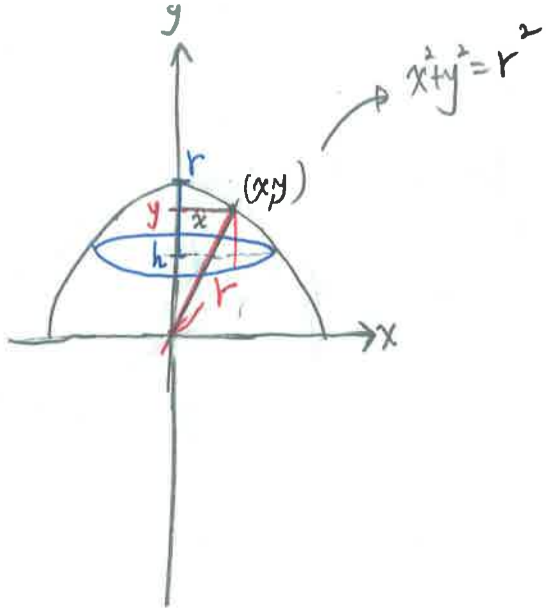
$$= 0-(-\sqrt{3})$$

$$= \sqrt{3}$$

\#

43.

$0<h<r$



$$V=\int_h^r \pi x^2\,dy=\int_h^r \pi(r^2-y^2)\,dy$$

$$=\pi\cdot\int_h^r r^2-y^2\,dy$$

$$=\pi\cdot\left(r^2y-\frac{1}{3}y^3\Big|_{y=h}^{y=r}\right)$$

$$=\pi\cdot\left[\frac{2}{3}r^3-\left(r^2h-\frac{1}{3}h^3\right)\right]$$

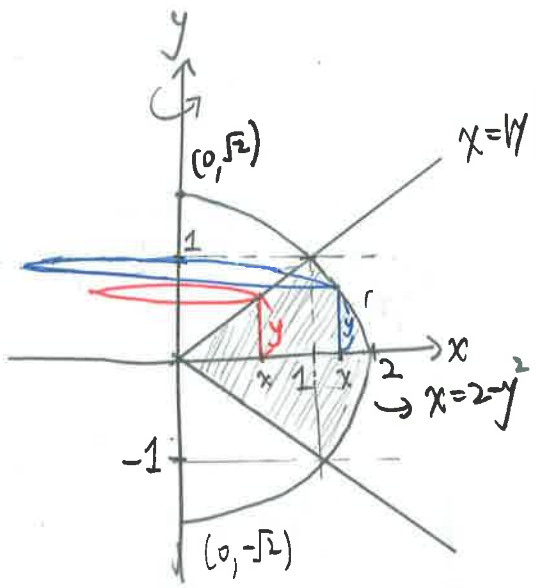
$$=\pi\cdot\left[\frac{2}{3}r^3-r^2h+\frac{1}{3}h^3\right]$$

※

5. Salas §6-3 ※ 12, 23, 39

12. $x=|y|$

$x=2-y^2$



(红色) (藍色)

$$V=\int_0^1 2\pi x\cdot 2y\,dx+\int_1^2 2\pi x\cdot 2y\,dx$$

$$V=\int_0^1 2\pi x\cdot 2x\,dx+\int_1^2 2\pi x\cdot 2\sqrt{2-x}\,dx$$

$$=4\pi\cdot\left(\frac{1}{3}x^3\Big|_{x=0}^{x=1}\right)+4\pi\cdot\int_1^2 x\cdot\sqrt{2-x}\,dx$$

(let $u=2-x$, $du=-dx$)

$$=\frac{4\pi}{3}-4\pi\cdot\int_1^0(2-u)\cdot\sqrt{u}\,du$$

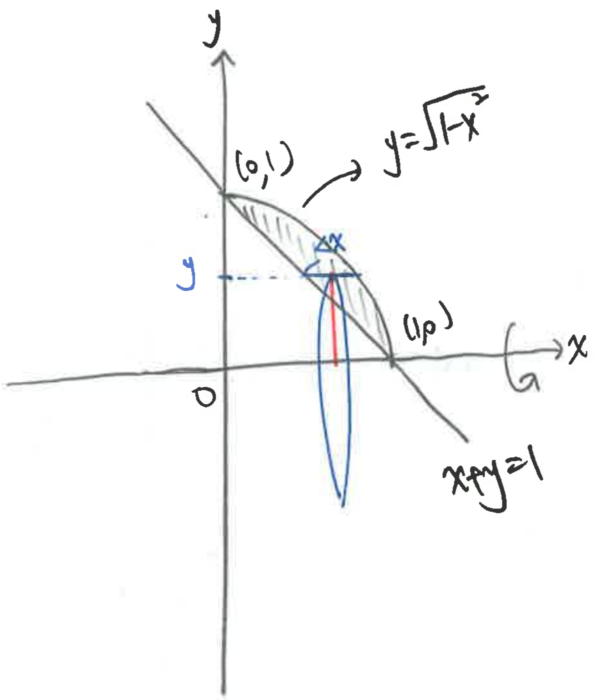
$$=\frac{4\pi}{3}+4\pi\cdot\int_0^1 2\sqrt{u}-u^{\frac{3}{2}}\,du$$

$$=\frac{4\pi}{3}+4\pi\cdot\left(\frac{4}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\Big|_{u=0}^{u=1}\right)$$

$$=\frac{4\pi}{3}+4\pi\cdot\frac{14}{15}=\frac{20\pi}{15}+\frac{56\pi}{15}=\frac{76}{15}\pi$$ ※

23. $y=\sqrt{1-x^2}$

$x+y=1$



$$V=\int_0^1 2\pi y\cdot\Delta x\,dy$$

$$=\int_0^1 2\pi y\cdot\left[\sqrt{1-y^2}-(1-y)\right]dy$$

$$=2\pi\cdot\int_0^1 y\sqrt{1-y^2}\,dy-2\pi\int_0^1 y-y^2\,dy$$

$\because \int_0^1 y-y^2\,dy=\frac{1}{2}y^2-\frac{1}{3}y^3\Big|_{y=0}^{y=1}=\frac{1}{6}$

$\because \int_0^1 y\sqrt{1-y^2}\,dy=\int_0^{\frac{\pi}{2}}\sin\theta\cdot\cos^2\theta\,d\theta=\frac{-1}{3}\cos^3\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}$

$=\frac{1}{3}$

$y=\sin\theta$

$dy=\cos\theta\,d\theta$

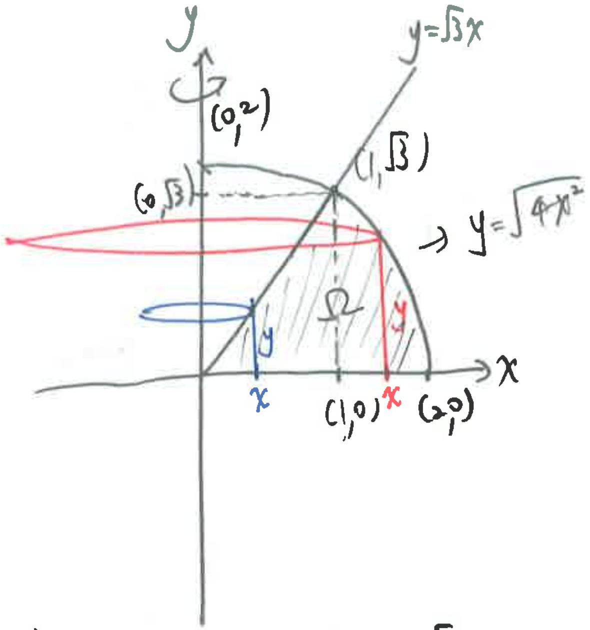
$0\le\theta\le\frac{\pi}{2}$

$\therefore V=2\pi\cdot\frac{1}{3}-2\pi\cdot\frac{1}{6}=\frac{1}{6}\cdot 2\pi=\frac{\pi}{3}$ ※

39.

$$f(x) = \begin{cases} \sqrt{3}x, & 0 \le x < 1 \\ \sqrt{4-x^2}, & 1 \le x \le 2 \end{cases}$$

(a)
$$V = \int_0^1 2\pi x \cdot y \, dx + \int_1^2 2\pi x \cdot y \, dx$$
$$= \int_0^1 2\pi x \cdot \sqrt{3}x \, dx + \int_1^2 2\pi x \cdot \sqrt{4-x^2} \, dx$$
$$= 2\sqrt{3}\pi \int_0^1 x^2 dx + 2\pi \int_1^2 x \cdot \sqrt{4-x^2} \, dx$$



$$\because \int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_{x=0}^{x=1} = \frac{1}{3}$$

$$\because \int_1^2 x \cdot \sqrt{4-x^2} \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin\theta \cdot 2\cos\theta \cdot 2\cos\theta \, d\theta$$

$x = 2\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$

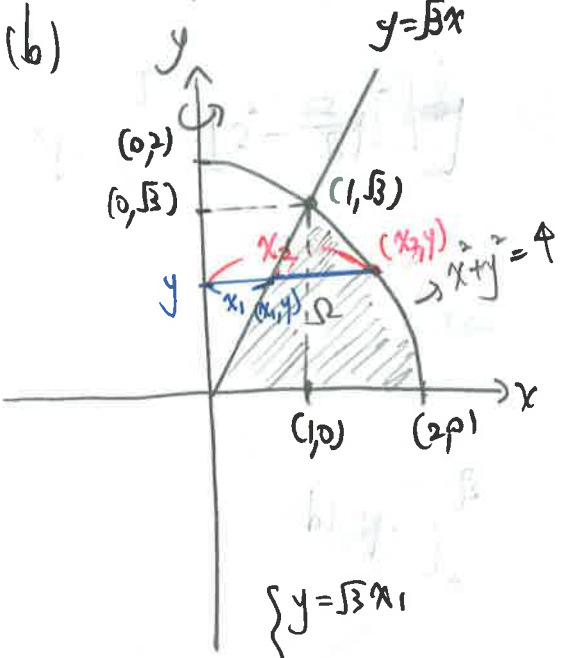
$dx = 2\cos\theta \, d\theta$

$$= 8 \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\theta \cdot \cos^2\theta \, d\theta$$
$$= 8 \cdot \left(-\frac{1}{3}\cos^3\theta \Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} \right) = \sqrt{3}$$

$$\therefore V = 2\sqrt{3}\pi \cdot \frac{1}{3} + 2\pi \cdot \sqrt{3} = \frac{8\sqrt{3}}{3}\pi$$

※

(b)



$$V = \int_0^{\sqrt{3}} (\pi x_2^2 - \pi x_1^2) \, dy = \int_0^{\sqrt{3}} \pi \left[4 - y^2 - \left(\frac{y}{\sqrt{3}}\right)^2\right] dy$$
$$= \int_0^{\sqrt{3}} \pi \left[4 - y^2 - \frac{y^2}{3}\right] dy = \int_0^{\sqrt{3}} \pi \left[4 - \frac{4}{3}y^2\right] dy$$
$$= \pi \cdot \left(4y - \frac{4}{9}y^3 \Big|_{y=0}^{y=\sqrt{3}} \right)$$

$$\begin{cases} y = \sqrt{3}x_1 \\ x_1 = \frac{y}{\sqrt{3}} \end{cases}$$

$$= \pi \cdot \left(4\sqrt{3} - \frac{4}{3}\sqrt{3} \right)$$

$$\begin{cases} x_2^2 + y^2 = 4 \\ x_2^2 = 4 - y^2 \end{cases}$$

$$= \frac{8\sqrt{3}}{3}\pi$$

※

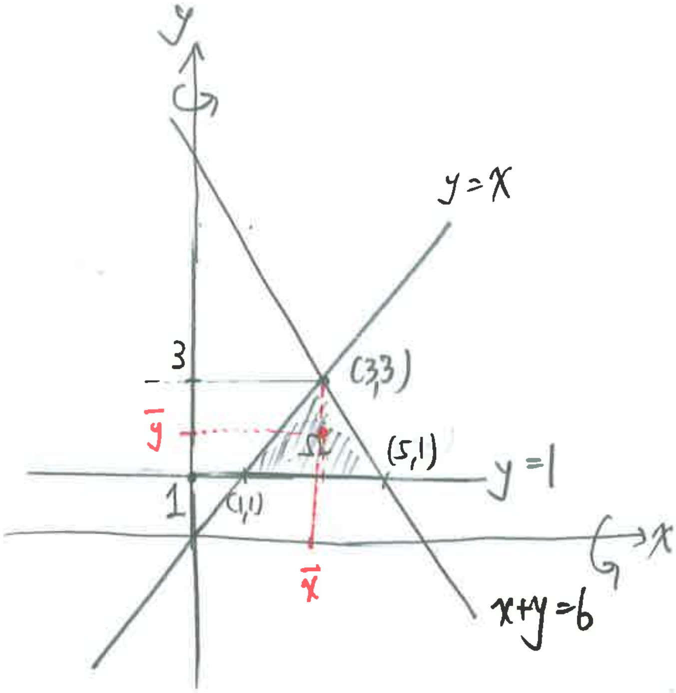
6. Salas §§6-4 : ※ 13, 25, 29, 30.

13.

$y=x$

$x+y=6$

$y=1$



$A = \text{area of } \Omega = \frac{1}{2} \times 4 \times 2 = 4$

By symmetry, then $\bar{x} = 3$.

$$\bar{y} \cdot A = \frac{1}{2} \cdot \int_1^3 x^2 - 1 \, dx + \frac{1}{2} \int_3^5 (6-x)^2 - 1 \, dx$$

$$= \frac{1}{2} \cdot \left[\frac{1}{3}x^3 - x \Big|_{x=1}^{x=3} \right] + \frac{1}{2} \cdot \left[\frac{1}{3}x^3 - 6x^2 + 35x \Big|_{x=3}^{x=5} \right]$$

$$= \frac{1}{2} \times \frac{20}{3} + \frac{1}{2} \times \frac{20}{3}$$

$$= \frac{20}{3} \Rightarrow \bar{y} \cdot 4 = \frac{20}{3} \Rightarrow \bar{y} = \frac{5}{3}$$

∴ Centroid is $(3, \frac{5}{3})$ ※

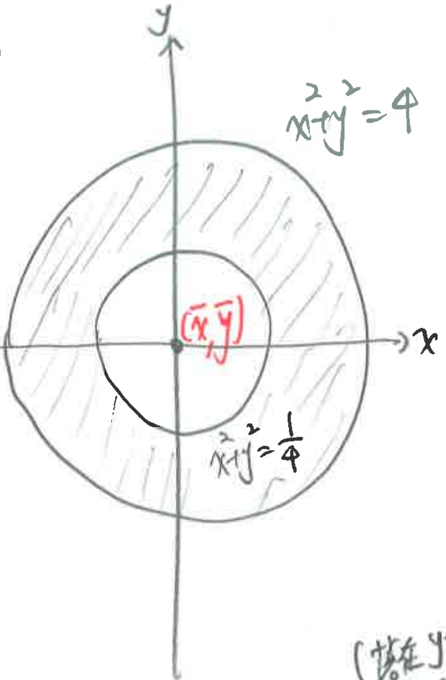
$V_x = 2\pi \bar{y} A = 2\pi \cdot \frac{5}{3} \cdot 4 = \frac{40}{3}\pi$ ※

$V_y = 2\pi \bar{x} A = 2\pi \cdot 3 \cdot 4 = 24\pi$ ※

$x^2+y^2=\frac{1}{4}$

$x^2+y^2=4$

(a)

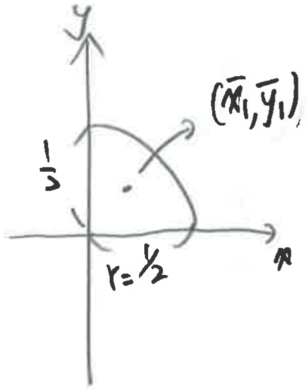


(a)

By symmetry, centroid is $(0,0)$ ※

(b) $\bar x=\bar y$

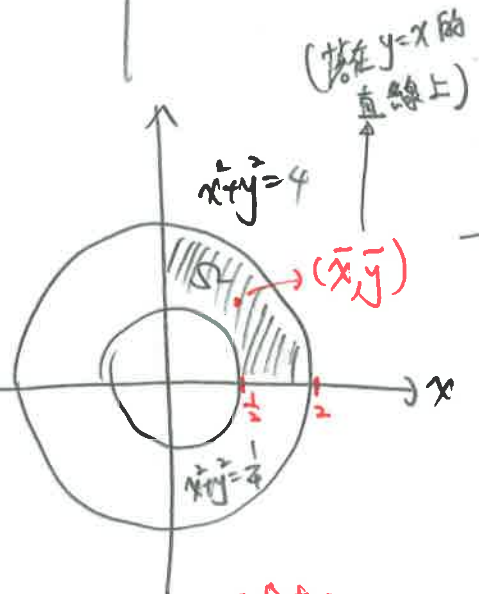
A = area of $\Omega = \pi-\frac{\pi}{16}=\frac{15}{16}\pi$

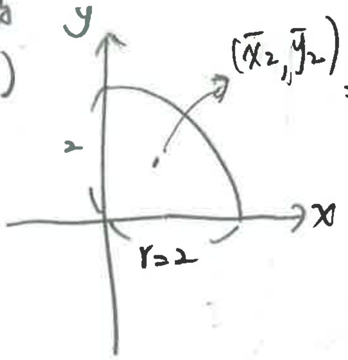


$\Rightarrow$ $\bar x_1=\bar y_1$ and area $=\frac{\pi}{16}$

$\bar y_1=\frac{4r}{3\pi}=\frac{2}{3\pi}$ ∴ $\bar x_1=\bar y_1=\frac{2}{3\pi}$

(b)





$\Rightarrow$ $\bar x_2=\bar y_2$ and area $=\pi$

$\bar y_2=\frac{4r}{3\pi}=\frac{8}{3\pi}$ ∴ $\bar x_2=\bar y_2=\frac{8}{3\pi}$

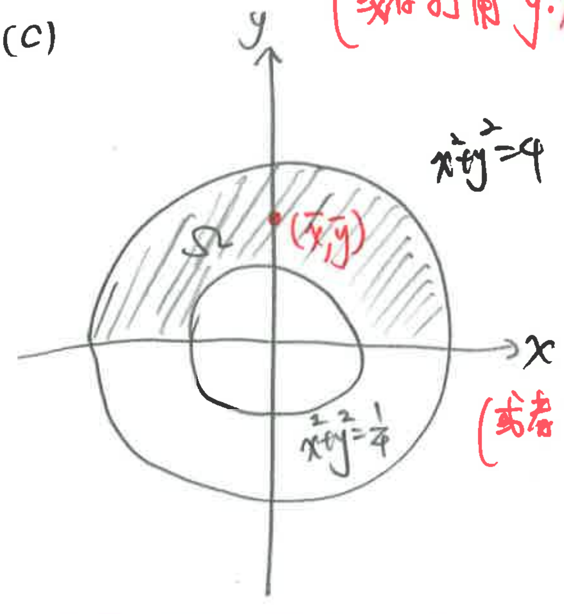
$\Rightarrow \bar x\cdot\frac{15}{16}\pi=\frac{8}{3\pi}\cdot\pi-\frac{2}{3\pi}\cdot\frac{\pi}{16}=\frac83-\frac{1}{24}=\frac{63}{24}=\frac{21}{8}$

$\Rightarrow \bar x=\frac{21}{8}\times\frac{16}{15\pi}=\frac{14}{5\pi}$ ∴ $\bar x=\bar y=\frac{14}{5\pi}$ ∴ $\left(\frac{14}{5\pi},\frac{14}{5\pi}\right)$ ※

(或者利用 $\bar y\cdot A=\frac12\int_0^{\frac12}(4-x^2)-(\frac14-x^2)\,dx+\frac12\int_{\frac12}^{2}(4-x^2)\,dx=\frac{21}{8}$ and $A=\frac{15}{16}\pi$

$\Rightarrow \bar y=\frac{14}{5\pi}$)

(c)



(c)

By symmetry, $\bar x=0$

∵(b) ∴ $\bar y=\frac{14}{5\pi}$ (指 (b) 的 centroid 向左搬到 y 軸上!!)

(或者利用 $\bar y\cdot A=\frac12\int_{-2}^{-\frac12}(4-x^2)\,dx+\frac12\int_{-\frac12}^{0}(4-x^2)-(\frac14-x^2)\,dx$

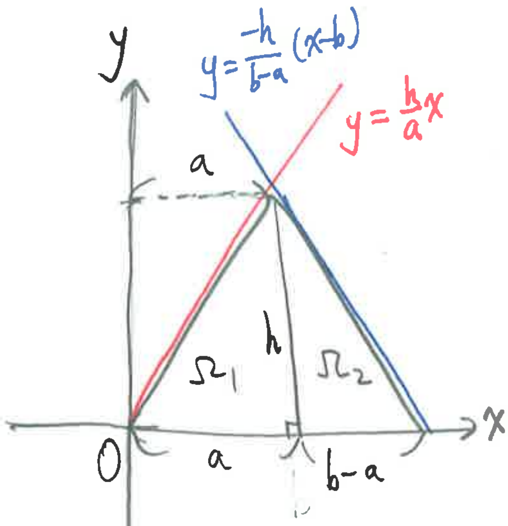
$+\frac12\int_0^{\frac12}(4-x^2)-(\frac14-x^2)\,dx+\frac12\int_{\frac12}^{2}(4-x^2)\,dx=\frac{21}{4}$ and $A=\frac{15}{8}\pi$，去求 $\bar y=\frac{14}{5\pi}$)

∴ $\left(0,\frac{14}{5\pi}\right)$ ※

29.1

(a)

area of $\Omega_1 = \frac{1}{2}ah$



$\Omega = \Omega_1 + \Omega_2$

$$\bar{x}\cdot\frac{1}{2}ah = \int_0^a x\cdot\frac{h}{a}x\,dx$$

$$= \frac{h}{a}\cdot\int_0^a x^2dx = \frac{h}{a}\cdot\frac{1}{3}a^3 = \frac{a^2h}{3}$$

$$\Rightarrow \bar{x} = \frac{2}{3}a$$

$$\bar{y}\cdot\frac{1}{2}ah = \frac{1}{2}\int_0^a \frac{h^2}{a^2}x^2dx = \frac{h^2}{2a^2}\int_0^a x^2dx$$

$$= \frac{h^2}{2a^2}\cdot\frac{1}{3}a^3 = \frac{ah^2}{6}$$

$$\Rightarrow \bar{y} = \frac{h}{3} \qquad \therefore (\bar{x},\bar{y}) = \left(\frac{2}{3}a, \frac{1}{3}h\right)$$

※

(b) area of $\Omega_2 = \frac{1}{2}(b-a)h$

$$\bar{x}\cdot\frac{1}{2}(b-a)\cdot h = \int_a^b x\cdot\frac{-h}{b-a}(x-b)\,dx = \frac{-h}{b-a}\int_a^b x^2-bx\,dx$$

$$= \frac{-h}{b-a}\left[\frac{1}{3}x^3-\frac{b}{2}x^2\Big|_{x=a}^{x=b}\right] = \frac{-h}{b-a}\cdot\left(\frac{-1}{6}b^3-\frac{1}{3}a^3+\frac{1}{2}a^2b\right)$$

$$\Rightarrow \bar{x} = \frac{-2}{(b-a)^2}\left(\frac{-1}{6}b^3-\frac{1}{3}a^3+\frac{1}{2}a^2b\right) = \frac{2}{3}a+\frac{1}{3}b$$

$$\bar{y}\cdot\frac{1}{2}(b-a)\cdot h = \frac{1}{2}\int_a^b \frac{h^2}{(b-a)^2}(x-b)^2dx = \frac{h^2}{2(b-a)^2}\int_a^b (x-b)^2dx$$

$$= \frac{h^2}{2(b-a)^2}\cdot\left[\frac{1}{3}(x-b)^3\Big|_{x=a}^{x=b}\right] = \frac{h^2}{2(b-a)^2}\cdot\frac{-(a-b)^3}{3}$$

$$\Rightarrow \bar{y} = \frac{2}{(b-a)h}\cdot\frac{h^2}{2(b-a)^2}\cdot\frac{-(a-b)^3}{3} = \frac{1}{3}h$$

$$\therefore (\bar{x},\bar{y}) = \left(\frac{2}{3}a+\frac{1}{3}b, \frac{1}{3}h\right)$$

※

(c)

area of $\Omega = \frac{1}{2}bh$

$$\bar{x}\cdot\frac{1}{2}bh = \int_0^a x\cdot\frac{h}{a}x\,dx + \int_a^b x\cdot\frac{-h}{b-a}(x-b)\,dx$$

$$= \frac{1}{3}a^2h - \frac{h}{b-a}\left(\frac{-1}{6}b^3 - \frac{1}{3}a^3 + \frac{1}{2}a^2b\right)$$

$$\Rightarrow \bar{x} = \frac{1}{3}a^2h\cdot\frac{2}{bh} - \frac{2}{bh}\cdot\frac{h}{b-a}\left(\frac{-1}{6}b^3 - \frac{1}{3}a^3 + \frac{1}{2}a^2b\right)$$

$$= \frac{2}{3}\cdot\frac{a^2}{b} + \frac{2}{b(b-a)}\left(\frac{b^3}{6} + \frac{a^3}{3} - \frac{a^2b}{2}\right)$$

$$= \frac{4a^2(b-a)}{6b(b-a)} + \frac{2\cdot(b^3+2a^3-3a^2b)}{6b(b-a)} = \frac{2b^3-2a^2b}{6(b-a)b} = \frac{2b(b^2-a^2)}{6b(b-a)}$$

$$= \frac{a+b}{3} = \frac{1}{3}a + \frac{1}{3}b$$

$$\bar{y}\cdot\frac{1}{2}bh = \int_0^a \frac{1}{2}\cdot\frac{h^2}{a^2}x^2\,dx + \int_a^b \frac{1}{2}\cdot\frac{h^2}{(b-a)^2}(x-b)^2\,dx$$

$$= \frac{ah^2}{6} - \frac{h^2(a-b)^3}{6(b-a)^2} = \frac{ah^2}{6} + \frac{h^2(b-a)}{6} = \frac{bh^2}{6}$$

$$\Rightarrow \bar{y} = \frac{2}{bh}\cdot\frac{bh^2}{6} = \frac{1}{3}h$$

$$\therefore (\bar{x},\bar{y}) = \left(\frac{1}{3}a + \frac{1}{3}b,\ \frac{1}{3}h\right)$$

30.

承29題，"整個三角形"的 centroid 為 $(\bar{x},\bar{y})=\left(\frac{1}{3}a+\frac{1}{3}b,\ \frac{1}{3}h\right)$

(a)

$$V_x=2\pi\bar{y}A=2\pi\cdot\frac{1}{3}h\cdot\frac{1}{2}bh=\frac{1}{3}\pi bh^2$$ ※

(b)

$$V_y=2\pi\bar{x}A=2\pi\cdot\left(\frac{1}{3}a+\frac{1}{3}b\right)\cdot\frac{1}{2}bh=\frac{1}{3}\pi(a+b)\cdot bh$$ ※