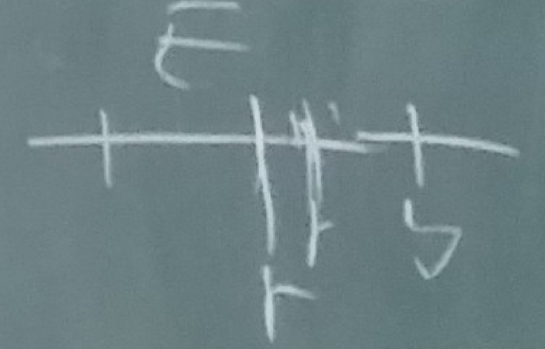
1. y is upper bound of E

Assume that y isn't $\sup E$, there exists a r is upper bound of E

and $r<y \Rightarrow \boxed{y \notin E}$ (why?)

Hence, if y is the upper bound of E,

and $y \in E \Rightarrow y$ must be $\sup E$



<pf> if $y \in E$

if $x<y$, then x isn't upper bound of E

exists a $z \in E$, s.t. $z > x$

let $z = y$, then x isn't upper bound of E

thus y is $\sup E$

2. If S has GLB, then S has LUB.

pf Suppose S has GLB.

Let $\phi \neq E$ be a bouned above subset of S

Let A be the set of upper bound for E

given $x \in E$. for all $a \in A$, a is an upper bound

By definition of upper bound, we have $x \leq a$

In particular, x is a lower bound for A

So. A is bounded below by x.

Since S has GLB. $\Rightarrow$ $\inf A$ exists

let $b = \inf A$.

Claim: $b = \sup E$

(i) b is an upper bound for E

Since every $y \in E$ is a lower bound for A

$\Rightarrow$ $b \geq y$, for each $y \in E$

$\Rightarrow$ b is an upper bound.

(ii) for all $r < b \Rightarrow r$ is Not an upper bound for E

$b = \inf A \Rightarrow \forall a \in A$, we have $a \geq b$

Hence if for all $r < b$, then we have $r \notin A$

$\Rightarrow r$ is Not an upper bound for E

Hence $\sup E$ exists $\Rightarrow$ S has LUB.

1. $r \in Q \quad x \notin Q$
$(r \neq 0)$

pf. Assume $r+x$ is rational
then $(r+x)-r = x$ must be rational
but $x \notin Q$ is contradiction
$\Rightarrow r+x$ is irrational

Assume rx is rational
then $\frac{rx}{r} = x$ $(r \neq 0)$ must be rational
but x is irrational is contradiction
$\Rightarrow rx$ is irrational

2. $\sqrt{12} \notin Q$
Assume $\frac{q}{p} \in Q$, $p, q \in Z$ and $(p, q) = 1$
then $\left(\frac{q}{p}\right)^2 = 12 \Rightarrow q^2 = 12p^2 = 3 \times 4p^2$
Let $q = 3k \Rightarrow 9k^2 = 12p^2 \Rightarrow 3k^2 = 4p^2$
$\Rightarrow \boxed{3 \mid p^2 \Rightarrow 3 \mid p} \Rightarrow (p, q) = 3$
is contradition
$\therefore$ do not exist $\left(\frac{q}{p}\right)^2 = 12$

if $3 \mid q^2$
then $3 \mid q$
(?)

証 $\inf A = -\sup(-A)$

Let A be a nonempty set of real number is bounded below.

Let $-A$ be the set of all numbers $-x$, where $x \in A$

Claim: $-\sup(-A)$ is greatest lower bound of A

Let $\alpha = -\sup(-A)$

Claim: ① $\forall x \in A \rightarrow \alpha \leq x$

② $\forall \gamma > \alpha \Rightarrow \gamma$ is not a lower bound of A

① $\forall x \in A \Rightarrow -x \in -A$

$\Rightarrow -x \leq \sup(-A)$

$\Rightarrow x \geq -\sup(-A) = \alpha$

α is a lower bound of A

② $\forall \gamma > \alpha \Rightarrow \gamma$ is not a lower bound of A

$\equiv \gamma$ is a lower bound of $A \Rightarrow \gamma \leq \alpha$

$\forall x \in A$ st. $\gamma \leq x \Rightarrow -x \leq -\gamma$

$\forall -x \in -A$ st. $-x \leq -\gamma$

$-\gamma$ is an upper bound of $-A$

Hence $-\gamma \geq \sup(-A)$

$\gamma \leq -\sup(-A) = \alpha$

$\Rightarrow -\sup(-A)$ is the greatest lower bound of A

ε-δ 證明 $\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$

$\forall \varepsilon > 0$, take $\delta = \underline{\min\{1, 2\varepsilon\}}$

s.t. $0<|x-1|<\delta \le 1$

$$\left|\frac{x}{x+1}-\frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} < \frac{|x-1|}{2\cdot 1} < \frac{\delta}{2} \le \varepsilon$$

估計 $\left|\frac{x}{x+1}-\frac{1}{2}\right| < \varepsilon$, if $0<|x-1|<\delta$

$$\left|\frac{x}{x+1}-\frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} < \frac{|x-1|}{2\cdot 1}$$

$$\boxed{\begin{aligned} &0<|x-1|<1 \\ &0<x<2 \\ &1<|x+1|<3 \end{aligned}}$$

6. If $f(x) \to L$ and $g(x) \to M \neq 0$ as $x \to c$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

pf

Claim: $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$

given $\varepsilon > 0$

Since $g(x) \to M \neq 0$ as $x \to c$

$\exists\ \delta_1 > 0$ s.t. $|g(x) - M| < \frac{|M|}{2}$ for $|x-c| < \delta_1$

$|M| = |M - g(x) + g(x)| \leq |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$

$\Rightarrow |g(x)| > \frac{|M|}{2} \Rightarrow \left|\frac{1}{g(x)}\right| < \frac{2}{|M|}$.

There exists $\delta_2 > 0$ s.t. $|g(x) - M| < \frac{|M|^2}{4}\varepsilon$, for $|x-c| < \delta_2$

Choose $\delta_3 = \min\{\delta_1, \delta_2\}$, for $|x-c| < \delta$.

We have $\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{g-M}{Mg}\right| = \frac{1}{|M|}\frac{1}{|g(x)|}|g(x)-M|$

$\leq \frac{1}{|M|}\frac{2}{|M|}\frac{M^2\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$.

Claim: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

By above, we have $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ and $\lim_{x\to c} f(x) = L$,

And we've known that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$\lim_{x\to c} f(x)\,g(x) = LM$

Hence $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} (f(x))\left(\frac{1}{g(x)}\right) = L\frac{1}{M} = \frac{L}{M}$ ▥